ミクロ・マクロ遷移と場の量子論

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はじめに

本研究課題は，1996 年度 ~1997 年度に科学研究費補助金（国際学術研究）のもとで遂行された研究課題「マクロ系及びメソスコピック系の場の量子論」を引き継いだものであり，近年注目を集めつつあるマクロ系あるいはメソスコピック系の挙動をミクロ・マクロ遷移という観点から統一的に捉え，これらの系に対する量子論の確立を目指したものである．特に，量子論の基本に関わる極めて重要な問題であるばかりでなく，ミクロ系とマクロ系との相互作用が本質的な役割を果たすと考えられている量子論的観測問題と関連する基本的諸問題を主要なテーマとして取り上げてきた．具体的研究項目及び成果・課題としては，

1. 量子系の時間発展の一般論と関連する諸問題を取り上げた．特に，多自由度系としての“環境系”のもとで量子系が時間発展する様子を，量子コヒーレンスの消滅過程，あるいは生存確率に対する崩壊則という観点から考察した．また，量子力学的観測過程に対する可解な力学的模型 (modified Coleman-Hepp 模型) の弱結合・マクロ極限で出現する確率過程 (Wiener 過程) に関しては，量子論的相関関数の物理的意味に考慮して分析を進めた．
2. 前項目とも関連して，中性子スピンを利用した量子ゼノン効果の干渉実験を，実際の実験的状況を考慮して分析した．特に，波束としての中性子の磁場領域通過における反射を考慮に入れると，最終的に得られる生存確率は理想的な場合とは全く異なる可能性のあることが判明した．反射によるロスを取り入れた場合に実際の実験結果と比較すべき数値がどうなるのか，引き続き具体的な評価を試みている．
3. Nelson 流の確率過程量子化を用いることによって，量子論での時間の問題を取り扱った．特に，いわゆるトンネル時間の評価では，振動型摂動ポテンシャルの印加によってどのような時間スケールが現われるのか，詳細な検討を加えている．また，この方法によって領域滞在時間を評価することにより，量子カオスへのアプローチが可能であるか試みた．
4. 古典系で良く知られた確率共鳴現象を量子系に適用するために，長時間・弱結合極限の一つである確率極限近似を用いて定式化を行った．特に，量子性が古典的確率共鳴現象にどのような影響を与え得るのかを調べるため，メソスコピック系での共鳴現象を詳しく調べた．また，確率極限近似の拡張を行って他の量子系（スピン・ボソン系）への適用を試みている．
5. 系の安定性・不安定性が議論できるような量子力学系として，非線形効果を持つ量子光学系を提案し，その理論的解析を進めるとともに実験的実現可能性を追究した．この系では，非線形効果による光子の指数関数的生成過程（不安定性）がそれに引き続く線形相互作用で抑制され得る（安定性）と理論的に予言されており，古典力学で有名なアーノルドの倒立振動子の量子力学版とも考えられ，大変興味深い．実験グループとも連絡を取りながら引き続き研究を進める予定である．

本研究課題の成果は，既に学会等で口頭発表されているだけでなく，論文として学術誌に公表されているものも多く，以下にその数編を掲載することで本研究課題の成果報告書に代

える．

本研究課題が目指してきた内容は，量子力学の基礎に関わるという意味において極めて幅広い内容を有しており，引き続き研究を進める予定である．

また，本研究課題は，(旧）国際学術研究でのバリ大学（イタリア）グループとの共同研究として始まったが，研究期間の2年間は言うに及ばず，これまでの共同研究は極めて実りの多いものであった．同グループとの共同研究は今後とも継続して行く予定である．

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TIME DEVELOPMENT OF A WAVE PACKET IN POTENTIAL SCATTERING

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ABSTRACT

A one-dimensional scattering problem off a δ-shaped potential is solved analytically and the time development of a wave packet is derived from the time-dependent Schrödinger equation. The exact and explicit expression of the scattered wave packet supplies us with interesting information about the "time delay" by potential scattering in the asymptotic region. It is demonstrated that a wave packet scattered by a spin-flipping potential can give us quite a different value for the delay times from that obtained without spin-degrees of freedom.

INTRODUCTION AND SUMMARY

It is well known that the "time" occupies a rather peculiar position in quantum mechanics in the sense that it is just a parameter in the Schrödinger equation and no satisfactory "time operator" has been found or devised so far, even though it is certainly measurable in experiments. For example, consider the definition of the interaction time of a particle with a potential when the particle is represented by an almost monochromatic wave packet and its spatial width is longer than the range of the potential. To let the interaction time be given by [interaction range]/[particle's velocity] is certainly based on the classical concept of the point-like particle and would be called in question in such a case. To treat this kind of problem, one sometimes uses the *time-independent* Schrödinger equation and forms a wave packet by superposing its solutions. Even though it can supply us with some information on the time development of the system under consideration, one has, at the same time, to remember in which sense this kind of treatment can represent the actual dynamical process: The solution to the time-independent Schrödinger equation is considered to describe the actual scattering process as an approximately stationary process, developing around the scatterer, where

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both the incoming and outgoing waves are present [1]. The use of stationary solutions is justified in this restricted sense. Of course, we may construct a wave packet from these solutions with an appropriate weight function, in order to see the explicit time dependence. In this case, however, we have to be careful about the choice of the very moment $t = 0$ since the actual scattering process is by no means translationally invariant (i.e. there is a moment when the particle is injected). In connection to this, we also need to correctly understand the meaning of the weight function. In this respect, it would be more natural and can be unambiguous to introduce a wave packet from the beginning to represent the incident particle and to examine its time evolution on the basis of the *time-dependent* Schrödinger equation. We shall set $t = 0$ when the particle is injected and the weight function is so chosen as to reproduce the incident wave packet.

The purpose of this work is twofold. First, a one-dimensional time-dependent Schrödinger equation for a particle scattered by a simple δ-shaped potential is solved analytically to give an exact and explicit wave-packet solution in section 2. The exact expression of the wave packet enables us to understand how its asymptotic behavior emerges as $t \to \infty$. The connection to the stationary solutions is easily seen. The analysis is extended to incorporate possible internal degrees of freedom in section 3. As a concrete and still solvable example, the scattering of a spin-1/2 particle off a spin-flipping δ-potential is considered and its wave-packet solution is derived.

Second, in section 4, we choose an almost monochromatic incident wave packet (i.e. spatially broad wave packet) and derive its asymptotic form, from which the mean position is calculated and compared with that of a free wave packet without interaction. In the simplest case with no internal degrees of freedom, it turns out that in spite of the vanishing interaction range of the δ-potential, the scattered wave packet exhibits a finite time delay or advance, depending on whether the interaction is repulsive or attractive. This delay (or advance) time is shown to agree with that derived from the energy derivative of the phase shift [2–4] caused by the potential scattering. In the spin-1/2 case, there are four scattering channels depending on whether the particle is transmitted or reflected and the spin is flipped or not. It is shown explicitly that contrary to the first case without spin-degrees of freedom, there remains no time delay or advance in any channels. This can be traced back to the reality of the scattering-matrix elements (i.e. the transmission and reflection coefficients) in this case. Notice that these two simple examples and their results imply that the notion of time delay of a wave packet can be quite different from the classical counterpart $\sim$ [interaction range]/[particle's velocity] and its estimation fully needs quantum mechanical treatment.

WAVE-PACKET SOLUTION FOR SCATTERING OFF A δ-POTENTIAL

Let us consider a simple scattering problem, i.e. a one-dimensional scattering of a particle off a single δ-potential. The Hamiltonian is given by

$$H = \frac{p^2}{2m} + g\delta(x). \tag{1}$$

We shall solve the time-dependent Schrödinger equation for a wave function in momentum space $\langle p|\psi\rangle_t$,

$$i\hbar\frac{\partial}{\partial t}\langle p|\psi\rangle_t = \langle p|H|\psi\rangle_t = \frac{p^2}{2m}\langle p|\psi\rangle_t + gC(t), \tag{2}$$

where we have put

$$C(t) \equiv \int \frac{dq}{2\pi\hbar}\langle q|\psi\rangle_t. \tag{3}$$

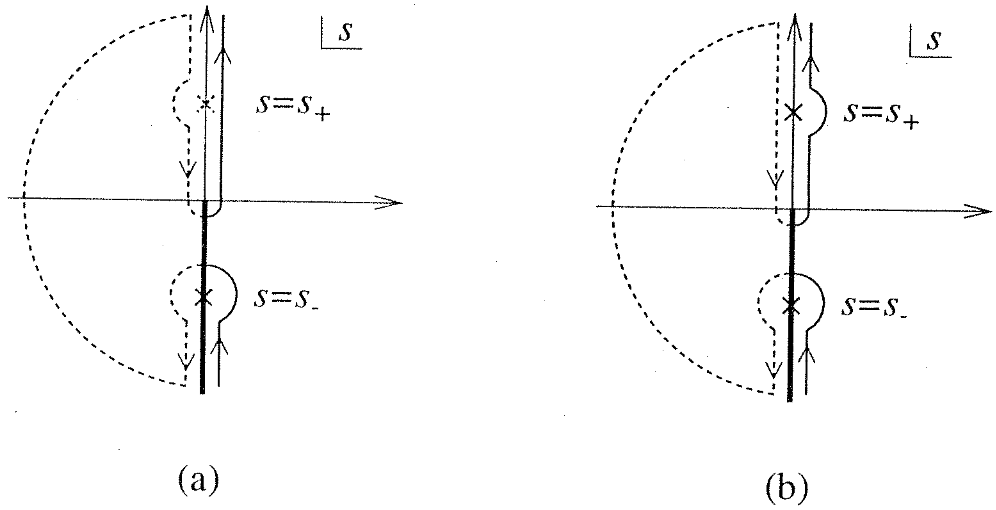

Figure 1. Singularities of the integrand in (7) and the deformed contours for (a) $g > 0$ and (b) $g < 0$. There are two simple poles, denoted by crosses, at $s_+ = img^2/2\hbar^3$ and $s_- = -ip^2/2m\hbar$ and a cut, represented by a bold line, along the negative imaginary s-axis in either case. Dashed lines run on the second Riemannian sheet.

The solution is easily found

$$\langle p|\psi\rangle_t = e^{-ip^2t/2m\hbar}\langle p|\psi\rangle_0 - i\frac{g}{\hbar}\int_0^t dt' e^{-ip^2(t-t')/2m\hbar}C(t') \tag{4}$$

in terms of the initial wave function $\langle p|\psi\rangle_0$ and the function C defined by (3). The Laplace transform of the latter $\tilde{C}(s) = \int_0^\infty dt e^{-st}C(t)$ $(s > 0)$ is easily found to be

$$\tilde{C}(s) = \frac{1}{1 + i(g/\hbar)\mathcal{F}(s)}\int \frac{dq}{2\pi\hbar}\frac{\langle q|\psi\rangle_0}{s + iq^2/2m\hbar}, \tag{5}$$

where

$$\mathcal{F}(s) = \int \frac{dq}{2\pi\hbar}\frac{1}{s + iq^2/2m\hbar} = \sqrt{\frac{m}{2\hbar}}\frac{1}{\sqrt{is}}. \tag{6}$$

In order to obtain $C(t)$ from $\tilde{C}(s)$ via the inverse Laplace transformation, we need to make an analytic continuation of the latter from the original domain of $s > 0$ into the left-half complex s-plane where $\mathrm{Re}\, s < 0$. Notice that the above function has a cut along the negative imaginary s-axis. We can write down $C(t')$ $(t' > 0)$ as

$$\begin{aligned} C(t') &= \int \frac{ds}{2\pi i} e^{st'}\tilde{C}(s) \\ &= \int \frac{dq}{2\pi\hbar}\langle q|\psi\rangle_0 \int_{-i\infty+\epsilon}^{i\infty+\epsilon} \frac{ds}{2\pi i}\frac{\sqrt{is}\{\sqrt{is} - i(g/\hbar)\sqrt{m/2\hbar}\}\, e^{st'}}{i(s - img^2/2\hbar^3)(s + iq^2/2m\hbar)}, \end{aligned} \tag{7}$$

where the integrand has a cut along the negative imaginary s-axis and two simple poles at $s = -iq^2/2m\hbar$ (for fixed q) and $s = img^2/2\hbar^3$, the latter of which exists only on the second Riemannian sheet if the coupling constant g is positive (repulsive potential) and on the first sheet if $g < 0$ (attractive potential). Therefore the original contour shall be deformed, avoiding these simple poles, to encircle in the left-half plane of the second Riemannian sheet (see Fig. 1).

The choice of these deformed contours facilitates the integration over s and we obtain

$$
\begin{aligned}
C(t') = \int \frac{dq}{2\pi\hbar}\langle q|\psi\rangle_0 \\
\times \Bigg[\frac{1}{1+img/\hbar|q|} e^{-iq^2t'/2m\hbar} + \frac{1}{1+(\hbar q/mg)^2} e^{img^2t'/2\hbar^3} \\
+ \frac{g}{\pi\hbar}\sqrt{\frac{m}{2\hbar}}\mathcal{P}\int_0^\infty dy \frac{\sqrt{y}e^{iyt'}}{(y-mg^2/2\hbar^3)(y+q^2/2m\hbar)} \Bigg],
\end{aligned}
\tag{8}
$$

which is valid irrespectively of the sign of g. Each term in the square parentheses in (8) corresponds to the contribution arising from each singularity of the integrand mentioned above.

This explicit expression for $C(t')$ leads us to an exact expression of the wave function in p-space [see Eq. (4)]

$$
\begin{aligned}
\langle p|\psi\rangle_t = e^{-ip^2t/2m\hbar} \\
\times \Bigg[\langle p|\psi\rangle_0 - i\frac{g}{\hbar}\int \frac{dq}{2\pi\hbar}\langle q|\psi\rangle_0 \Bigg\{ \frac{1}{1+img/\hbar|q|}\Delta_t[(p^2-q^2)/2m\hbar] \\
+ \frac{1}{1+(\hbar q/mg)^2}\Delta_t[(p^2+(mg/\hbar)^2)/2m\hbar] \\
+ \frac{g}{\pi\hbar}\sqrt{\frac{m}{2\hbar}}\mathcal{P}\int_0^\infty dy \frac{\sqrt{y}\,\Delta_t[y+p^2/2m\hbar]}{(y-mg^2/2\hbar^3)(y+q^2/2m\hbar)} \Bigg\} \Bigg],
\end{aligned}
\tag{9}
$$

where

$$\Delta_t[x] \equiv \frac{e^{ixt}-1}{ix}. \tag{10}$$

Notice that this is an exact expression [5,6] for the wave function at $t > 0$ and no assumption has been made in its derivation.

Obviously the first term represents the free evolution of the incident wave packet, while the other terms stand for the scattered components. It would be interesting to observe that apart from the factor $\exp(-ip^2t/2m\hbar)$, the t dependence appears only through the functions Δ_ts, each of which contains an energy-nonconserving component. The appearance of these energy-nonconserving terms is due to the energy-time uncertainty relation and they are expected to die out when the wave packet has passed through the interaction region and reaches the so-called wave zone. In order to understand this phenomenon clearly, let us consider the asymptotic limit $t \to \infty$ in (9).

It is clear that in the asymptotic region $t \to \infty$, the wave packet $\langle p|\psi\rangle_t$ is confined to the energy shell, since the function $\Delta_t[x]$ approaches the Dirac δ-function in this limit

$$\Delta_t[x] \stackrel{t\to\infty}{\longrightarrow} 2\pi\delta(x), \tag{11}$$

and is given by

$$
\begin{aligned}
\langle p|\psi\rangle_t &\stackrel{t\to\infty}{\longrightarrow} e^{-ip^2t/2m\hbar}\left\{ \langle p|\psi\rangle_0 - i\frac{mg}{\hbar|p|}\frac{1}{1+img/\hbar|p|}\Big(\langle p|\psi\rangle_0 + \langle -p|\psi\rangle_0\Big) \right\} \\
&= e^{-ip^2t/2m\hbar}\left\{ \frac{1}{1+img/\hbar|p|}\langle p|\psi\rangle_0 + \frac{-img/\hbar|p|}{1+img/\hbar|p|}\langle -p|\psi\rangle_0 \right\}.
\end{aligned}
\tag{12}
$$

Observe that the two terms in the last expression correspond to the transmitted and reflected waves and the former is given by the sum of the free (nonscattered) and the scattered waves. Eq. (12) shows clearly and explicitly the connection of the wave-packet (i.e. the time-dependent) solution to the stationary solution in the asymptotic region: In fact, the two factors in front of $\langle p|\psi\rangle_0$ and $\langle -p|\psi\rangle_0$ are nothing but the transmission ($\mathcal{T}$) and reflection ($\mathcal{R}$) coefficients, respectively, obtained for plane-wave scattering with definite momentum p

$$\mathcal{T}(p) = \frac{1}{1+img/\hbar|p|}, \qquad \mathcal{R}(p) = \frac{-img/\hbar|p|}{1+img/\hbar|p|}. \tag{13}$$

SCATTERING OFF A SPIN-FLIPPING δ-POTENTIAL AND ITS WAVE-PACKET SOLUTION

Let a spin-1/2 particle prepared in the up-state be injected at $t=0$ to the spin-flipping δ-potential. Two spin-eigenstates of the particle, energetically separated by $\hbar\omega$, shall be taken to be those of the third Pauli matrix σ_3, so that the total Hamiltonian is

$$H = \frac{p^2}{2m} + \frac{\hbar\omega}{2}(1+\sigma_3) + g\delta(x)\sigma_1. \tag{14}$$

It is important to note that here we retain the spatial degrees of freedom of the particle. This is in contrast to the case in which, e.g. spin flip by a magnetic field is considered, where the spatial degrees of freedom are completely neglected and the interaction time is set to be [range of magnetic field]/[particle's velocity] from the outset. This is justifiable under the situation where the range of interaction is much larger than the width of the wave packet representing the incident particle, even if the latter is broad enough to approximate the particle as a plane wave. Obviously this condition does not hold in our case, since the interaction range of the δ-potential is 0. Actually, a question about the interaction time for such a short-range potential is one of the motivations of the present work.

If we decompose the wave function in terms of momentum and spin eigenstates

$$|\psi\rangle_t = \int dp\,[|p\uparrow\rangle\langle p\uparrow|\psi\rangle_t + |p\downarrow\rangle\langle p\downarrow|\psi\rangle_t], \tag{15}$$

the Schrödinger equation becomes the coupled equations between $\langle p\uparrow|\psi\rangle_t$ and $\langle p\downarrow|\psi\rangle_t$

$$i\hbar\frac{\partial}{\partial t}\langle p\uparrow|\psi\rangle_t = \Big(\frac{p^2}{2m}+\hbar\omega\Big)\langle p\uparrow|\psi\rangle_t + gC_\downarrow(t), \tag{16}$$

$$i\hbar\frac{\partial}{\partial t}\langle p\downarrow|\psi\rangle_t = \frac{p^2}{2m}\langle p\downarrow|\psi\rangle_t + gC_\uparrow(t), \tag{17}$$

where

$$C_\uparrow(t) \equiv \int\frac{dq}{2\pi\hbar}\langle q\uparrow|\psi\rangle_t, \qquad C_\downarrow(t) \equiv \int\frac{dq}{2\pi\hbar}\langle q\downarrow|\psi\rangle_t. \tag{18}$$

Since the initial state is set in a spin-up state, $\langle p\downarrow|\psi\rangle_0 = 0$ and the solution to Eqs. (16) and (17) is given by

$$\langle p\uparrow|\psi\rangle_t = e^{-i(p^2/2m\hbar+\omega)t}\langle p\uparrow|\psi\rangle_0 - i\frac{g}{\hbar}\int_0^t dt'\,e^{-i(p^2/2m\hbar+\omega)(t-t')}C_\downarrow(t'), \tag{19}$$

$$\langle p\downarrow|\psi\rangle_t = -i\frac{g}{\hbar}\int_0^t dt'\,e^{-ip^2(t-t')/2m\hbar}C_\uparrow(t'). \tag{20}$$

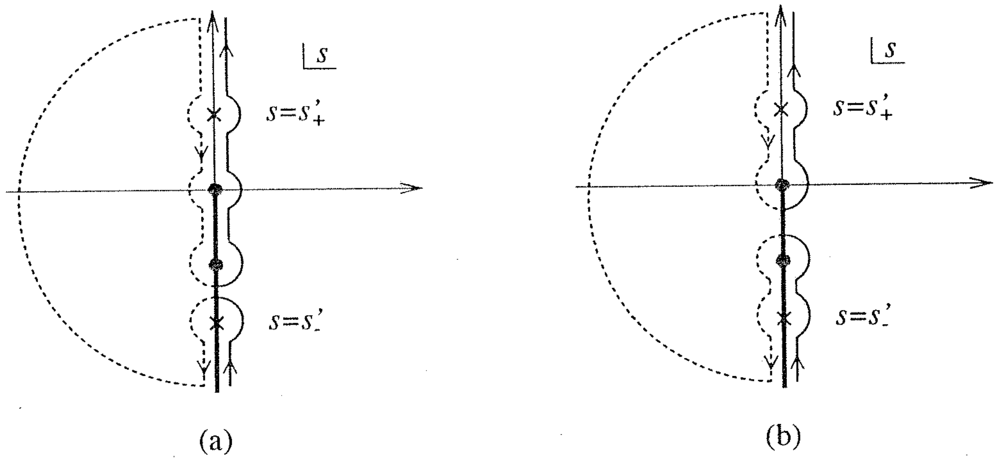

Figure 2. Deformed contours for the evaluation of (a) $C_\downarrow(t')$ and (b) $C_\uparrow(t')$. There are two simple poles, denoted by crosses, at $s'_+ = i(\sqrt{\omega^2+(mg^2/\hbar^3)^2}-\omega)/2$ and $s'_- = -i(p^2/2m\hbar+\omega)$, two branch points, denoted by dots, at $s=0$ and $s=-i\omega$ and a cut, represented by a bold line, along the negative imaginary s-axis. Dashed lines run on the second Riemannian sheet.

After an elementary calculation we find the solution

$$\begin{bmatrix}\tilde{C}_\uparrow(s)\\ \tilde{C}_\downarrow(s)\end{bmatrix} = \int \frac{dq}{2\pi\hbar}\frac{\langle q\uparrow|\psi\rangle_0}{s+i(q^2/2m\hbar+\omega)}\frac{\sqrt{s(s+i\omega)}}{\sqrt{s(s+i\omega)}-(img^2/2\hbar^3)}\begin{bmatrix}1\\ -(g/\hbar)\sqrt{im/2\hbar s}\end{bmatrix}. \tag{21}$$

It is clear that the Laplace transforms $\tilde{C}_\uparrow(s)$ and $\tilde{C}_\downarrow(s)$ have two simple poles at $s = i(\sqrt{\omega^2+(mg^2/\hbar^3)^2}-\omega)/2$ and $s=-i(q^2/2m\hbar+\omega)$, two branch points at $s=0$ and $s=-i\omega$ and a cut along the negative imaginary s-axis. The final expression of $C_\uparrow(t')$ and $C_\downarrow(t')$ is dependent on how the integration contour is deformed in the complex s-plane.

For the evaluation of $C_\downarrow(t')$, we shall deform the contour as in Fig. 2(a) to obtain

$$\begin{aligned} C_\downarrow(t') = \int \frac{dq}{2\pi\hbar}&\langle q\uparrow|\psi\rangle_0 \\ \times\Bigg[&-i\frac{mg}{\hbar}e^{-i(q^2/2m\hbar+\omega)t'}\frac{\sqrt{q^2(q^2+2m\hbar\omega)}}{\sqrt{q^2(q^2+2m\hbar\omega)}+(mg/\hbar)^2} \\ &+i\frac{g}{\hbar}\sqrt{\frac{m}{2\hbar}}\mathcal{P}\int_0^\infty\frac{dy}{\pi}\frac{e^{iyt'}}{y+(q^2/2m\hbar+\omega)}\frac{\sqrt{y+\omega}}{\sqrt{y(y+\omega)}-(mg^2/2\hbar^3)} \\ &+i\sqrt{\frac{m^3}{2\hbar^3}}\Big(\frac{g}{\hbar}\Big)^3\int_0^\omega\frac{dy}{2\pi}\frac{e^{-iyt'}}{y-(q^2/2m\hbar+\omega)}\frac{\sqrt{\omega-y}}{y(\omega-y)+(mg^2/2\hbar^3)^2}\Bigg]. \end{aligned} \tag{22}$$

Inserting this into (19) and integrating over t', we reach an exact formula for $\langle p\uparrow|\psi\rangle_t$. In particular, in the asymptotic region $t\to\infty$, only the first term can survive to give the asymptotic behavior of $\langle p\uparrow|\psi\rangle_t$

$$\langle p\uparrow|\psi\rangle_t \overset{t\to\infty}{\longrightarrow} e^{-i(p^2/2m\hbar+\omega)t}\langle p\uparrow|\psi\rangle_0$$

$$
+\frac{mg^2}{\hbar^2}\int\frac{dq}{2\pi\hbar}\langle q\uparrow|\psi\rangle_0\frac{\sqrt{q^2}\,e^{-i(p^2/2m\hbar+\omega)t}}{\sqrt{q^2(q^2+2m\hbar\omega)+(mg/\hbar)^2}}4\pi m\hbar\delta(p^2-q^2)
$$

$$
=e^{-i(p^2/2m\hbar+\omega)t}\Bigg[\frac{\sqrt{p^2(p^2+2m\hbar\omega)}}{\sqrt{p^2(p^2+2m\hbar\omega)+(mg/\hbar)^2}}\langle p\uparrow|\psi\rangle_0
+\frac{-(mg/\hbar)^2}{\sqrt{p^2(p^2+2m\hbar\omega)+(mg/\hbar)^2}}\langle -p\uparrow|\psi\rangle_0\Bigg]. \tag{23}
$$

The first and second terms in the parentheses respectively stand for the components, transmitted and reflected through the potential without changing the spin state and the factors multiplying them coincide with the ordinary transmission ($\mathcal{T}_\uparrow$) and reflection ($\mathcal{R}_\uparrow$) coefficients obtained by solving the time-independent Schrödinger equation. Observe that these coefficients are both real quantities.

Similar treatment can be done for the evaluation of $C_\uparrow(t')$: We shall take the deformed contour, depicted in Fig. 2(b), and get

$$
C_\uparrow(t')=\int\frac{dq}{2\pi\hbar}\langle q\uparrow|\psi\rangle_0\Bigg[\frac{q^2(q^2+2m\hbar\omega)}{q^2(q^2+2m\hbar\omega)-(mg/\hbar)^4}e^{-i(q^2/2m\hbar+\omega)t'}
-i\frac{mg^2}{\hbar^3}\mathcal{P}\int_\omega^\infty\frac{dy}{2\pi}\frac{e^{-iyt'}}{y-(q^2/2m\hbar+\omega)}\frac{\sqrt{y(y-\omega)}}{y(y-\omega)-(mg^2/2\hbar^3)^2}\Bigg]. \tag{24}
$$

In the $t\to\infty$ limit, $\langle p\downarrow|\psi\rangle_t$ is shown to behave like

$$
\langle p\downarrow|\psi\rangle_t\overset{t\to\infty}{\longrightarrow}-i\frac{mg}{\hbar}\theta\Big[\frac{p^2}{2m}-\hbar\omega\Big]e^{-ip^2t/2m\hbar}
\times\Bigg[\frac{\sqrt{p^2}}{\sqrt{p^2(p^2-2m\hbar\omega)+(mg/\hbar)^2}}\langle\sqrt{p^2-2m\hbar\omega}\uparrow|\psi\rangle_0
+\frac{\sqrt{p^2}}{\sqrt{p^2(p^2-2m\hbar\omega)+(mg/\hbar)^2}}\langle-\sqrt{p^2-2m\hbar\omega}\uparrow|\psi\rangle_0\Bigg]. \tag{25}
$$

These two terms correspond to the transmitted and reflected components of the particle, whose spin state is flipped to the down state through the interaction with the potential. The connection to the plane-wave solution is clear, since the common factor in the parentheses coincides with the transmission ($\mathcal{T}_\downarrow$) and reflection ($\mathcal{R}_\downarrow$) coefficients, which are both real and take the same form in this particular case. Observe the presence of the θ-function, which is necessary for energy conservation, since the total energy of the particle in the down state is given by $p^2/2m$ and the particle will gain energy $\hbar\omega$ by flipping its spin from up to down.

ESTIMATION OF THE DELAY TIME

The exact solutions to the time-dependent Schrödinger equations we have obtained in the previous sections explicitly show that their wave-packet solutions converge in the asymptotic ($t\to\infty$) limit to the states composed of the transmitted and reflected waves, multiplied by the relevant transmission ($\mathcal{T}$) and reflection ($\mathcal{R}$) coefficients for the plane wave solution

$$\langle p|\psi\rangle_t \stackrel{t\to\infty}{\longrightarrow} e^{-iE_pt/\hbar}\left[\mathcal{T}(p)\langle p|\psi\rangle_0 + \mathcal{R}(p)\langle -p|\psi\rangle_0\right], \tag{26}$$

where E_p is the energy of the state $|p\rangle$. For the moment, for notational simplicity, the dependence on the possible internal degrees of freedom of the particle shall be suppressed.

In order to illustrate the asymptotic behavior of the wave packet more clearly and quantitatively, let us express the above wave function in ordinary configuration space. We assume that the initial particle is represented by an almost monochromatic wave packet with a Gaussian profile

$$\langle p|\psi\rangle_0 = \mathcal{N}\exp\Big[-\frac{(p-p_0)^2}{4\delta p^2} - \frac{i}{\hbar}(p-p_0)x_0\Big], \tag{27}$$

where $\mathcal{N}$ is a normalization constant and $\delta p \ll p_0$. This wave packet is distributed around its mean position $x_0 < 0$ with width $\hbar/2\delta p$ and moves with an average momentum $p_0 > 0$ toward the potential. In order not to make any sensible overlap between the initial wave packet and the potential located at the origin, the inequality $x_0 + \hbar/2\delta p \ll 0$ is also assumed.

The transmitted wave packet in x-space is given by the Fourier transform of the first term of the RHS of Eq. (26)

$$\langle x|\psi\rangle_t^{\rm tr} = \int \frac{dp}{\sqrt{2\pi\hbar}}\mathcal{T}(p)e^{-iE_pt/\hbar+ipx/\hbar}\langle p|\psi\rangle_0. \tag{28}$$

If the transmission coefficient $\mathcal{T}(p)$ is a slowly varying function of p, we may expand $\mathcal{T}(p)$ around p_0 as a power series of $p-p_0$ to perform a Gaussian integral over p in (28). Under this assumption, the transmitted wave packet (in the asymptotic region) turns out to take also the Gaussian form

$$\langle x|\psi\rangle_t^{\rm tr} \sim \mathcal{T}(p_0)\exp\Big[-\frac{1}{4\hbar^2 u}\Big(x-x_0-p_0t/m-i\hbar\mathcal{T}'(p_0)/\mathcal{T}(p_0)\Big)^2+\frac{i}{\hbar}p_0x-\frac{i}{\hbar}E_{p_0}t\Big], \tag{29}$$

where

$$u = \frac{1}{4\delta p^2} + \frac{it}{2m\hbar}. \tag{30}$$

It is straightforward, if lengthy, to extract the real part of the exponent in (29), which can be written in the following form

$$-\frac{\delta p^2}{\hbar^2}\frac{1}{1+(4E_{\delta p}t/\hbar)^2}\Big[x-x_0-v_{\rm tr}(t-\delta_{\rm tr})\Big]^2 + (\delta p^2/p_0^2)\Big[p_0{\rm Re}\Big(\mathcal{T}'(p_0)/\mathcal{T}(p_0)\Big)\Big]^2, \tag{31}$$

where $v_{\rm tr}$ stands for the mean velocity of the transmitted wave packet

$$v_{\rm tr} = \frac{p_0}{m}\Big[1+2(\delta p^2/p_0^2)p_0{\rm Re}\Big(\mathcal{T}'(p_0)/\mathcal{T}(p_0)\Big)\Big]. \tag{32}$$

The quantity $\delta_{\rm tr}$ in (31), given by

$$\delta_{\rm tr} = \frac{m\hbar}{p_0}{\rm Im}\Big(\mathcal{T}'(p_0)/\mathcal{T}(p_0)\Big)\Big[1+2(\delta p^2/p_0^2)p_0{\rm Re}\Big(\mathcal{T}'(p_0)/\mathcal{T}(p_0)\Big)\Big]^{-1}, \tag{33}$$

measures a temporal displacement of the transmitted wave packet and is called the time delay or advance, according to its sign. It is interesting to note that even though our derivation is different from that in [2–4], the main part of the above $\delta_{\rm tr}$ coincides with that obtained by taking the energy derivative of the scattering phase shift [2–4]: The correction, however, is due to the wave packet effect, which is obtainable only when the wave packet itself is treated directly.

Similarly, we shall define the reflected wave packet (in the asymptotic region) by

$$\langle x|\psi\rangle_t^{\rm rf} = \int \frac{dp}{\sqrt{2\pi\hbar}} \mathcal{R}(p) e^{-iE_p t/\hbar + ipx/\hbar} \langle -p|\psi\rangle_0 \tag{34}$$

and perform a Gaussian integral over p, under similar conditions as in (29). The behavior of the reflected wave packet, again shown to be a Gaussian, is read from the real part of the exponent

$$-\frac{\delta p^2}{\hbar^2}\frac{1}{1+(4E_{\delta p}t/\hbar)^2}\left[x + x_0 + v_{\rm rf}(t-\delta_{\rm rf})\right]^2 + (\delta p^2/p_0^2)\left[p_0 {\rm Re}\big(\mathcal{R}'(p_0)/\mathcal{R}(p_0)\big)\right]^2, \tag{35}$$

from which we obtain its mean velocity and the time delay

$$v_{\rm rf} = \frac{p_0}{m}\left[1 + 2(\delta p^2/p_0^2) p_0 {\rm Re}\big(\mathcal{R}'(p_0)/\mathcal{R}(p_0)\big)\right], \tag{36}$$

$$\delta_{\rm rf} = \frac{m\hbar}{p_0}{\rm Im}\big(\mathcal{R}'(p_0)/\mathcal{R}(p_0)\big)\left[1 + 2(\delta p^2/p_0^2) p_0 {\rm Re}\big(\mathcal{R}'(p_0)/\mathcal{R}(p_0)\big)\right]^{-1}. \tag{37}$$

It would be interesting to estimate the value of the delay times for simple potential scatterings we have considered in the previous sections. Since the transmission and reflection coefficients for a particle with a definite momentum p_0, scattered by the δ-potential given in (1), are given by

$$\mathcal{T}(p_0) = \frac{1}{1+i\Omega(p_0)}, \qquad \mathcal{R}(p_0) = \frac{-i\Omega(p_0)}{1+i\Omega(p_0)}, \tag{38}$$

where $\Omega(p_0) \equiv mg/\hbar p_0$, the delay times for the transmitted and reflected wave packets are estimated to be

$$\delta_{\rm tr} = \frac{\hbar}{2}\frac{\Omega(p_0)}{E_{p_0}}|\mathcal{T}(p_0)|^2\left[1 + 2(\delta p^2/p_0^2)|\mathcal{R}(p_0)|^2\right]^{-1}, \tag{39}$$

$$\delta_{\rm rf} = \frac{\hbar}{2}\frac{\Omega(p_0)}{E_{p_0}}|\mathcal{T}(p_0)|^2\left[1 - 2(\delta p^2/p_0^2)|\mathcal{T}(p_0)|^2\right]^{-1}. \tag{40}$$

Notice that both delay times $\delta_{\rm tr}$ and $\delta_{\rm rf}$ are positive, showing time delays of the wave packets, for a repulsive potential ($g > 0$), while, for an attractive one ($g < 0$), they become negative, which imply that the wave packets are actually advanced. Observe also that these time delays vanish not only for the weak coupling limit $g \to 0$, but also for the strong coupling limit $g \to \infty$. Because in either limit, the wave packet is completely transmitted or reflected, we may understand that for the wave packet to exhibit a finite time delay or advance, the presence of both waves, transmitted and reflected, is crucial and that this phenomenon is ascribable to a kind of interference effect. This is still at a speculative level, however, we can say that the time-delay phenomena have an essentially quantum origin and can not be compared with the classical counterpart, which would also be clear from the fact that the δ-potential has a vanishing range of interaction.

The nontrivial nature of the time delay may be seen in the other case considered in section 3., where a scattering of a spin-1/2 particle off the spin-flipping δ-potential has been analyzed. We have shown explicitly in (23) and (25) that the scattered wave packet in each channel is represented just like (26) with channel-dependent transmission and reflection coefficients. If a particle prepared in the up state with definite momentum p_0 is scattered by the spin-flipping potential given in (14), these coefficients read

$$\begin{pmatrix} \mathcal{T}_\uparrow(p_0) \\ \mathcal{T}_\downarrow(p_0) \end{pmatrix} = \frac{\sqrt{p_0^2}}{\sqrt{p_0^2(p_0^2 + 2m\hbar\omega) + (mg/\hbar)^2}} \begin{pmatrix} \sqrt{p_0^2 + 2m\hbar\omega} \\ -img/\hbar \end{pmatrix}, \tag{41}$$

$$\begin{pmatrix} \mathcal{R}_\uparrow(p_0) \\ \mathcal{R}_\downarrow(p_0) \end{pmatrix} = -i\frac{mg}{\hbar}\frac{1}{\sqrt{p_0^2(p_0^2 + 2m\hbar\omega) + (mg/\hbar)^2}} \begin{pmatrix} -img/\hbar \\ \sqrt{p_0^2} \end{pmatrix}. \tag{42}$$

Notice that these quantities are essentially real and therefore no finite delay times are expected for any channels. [See Eqs. (33) and (37).]

Even though the appearance or disappearance of such delay times in potential scatterings has yet to be understood well, we can conclude that its analysis certainly requires completely quantum mechanical treatment.

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TIME SYMMETRY AND QUANTUM DEPHASING

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ABSTRACT

We first stress that the time symmetry in quantum mechanics manifests itself in the analytical properties of the Fourier transform of the evolution operator in the complex E-plane (E being the variable conjugate to time), in such a way that all singularities are distributed only on the real axis in the first Riemannian sheet, and new poles appear, in the N infinite limit (N standing for the number of degrees of freedom of detector or instrument concerned), on the second Riemannian sheet in a symmetric way with respect to the real axis. We then examine the symmetry-breaking phenomenon, such as decay or dissipation, by setting up the initial value problem: The temporal evolution of the transition probability is divided into three parts, the first being Gaussian for very short times, the second exponential for intermediate times and the third of the power type for very long times. We know that the Gaussian decay is directly connected to the so-called quantum Zeno effect, the exponential decay corresponds to a sort of dephasing process, because the time rate of the total transition probability becomes a sum of time rates of partial probabilities, and both the Gaussian-like and power-like decay will disappear, leaving only the exponential one, in the van Hove limit. The dominance of the exponential decay is equivalent to the appearance of a master equation, which tells us that we have no phase-correlation but decoherence or dephasing. All temporal behaviors of quantum-mechanical transition probability and related physics are reflected in the analytical property of the Fourier transform of the evolution operator.

INTRODUCTION

Needless to say, the time symmetry (or the time-reversal invariance) is one of the most important symmetries in physics. On the other hand, we know that its

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symmetry breaking in quantum mechanical transition probabilities is observed in decay or dissipation processes.

In this paper we first formulate the time symmetry in terms of analytical property of the Fourier transform of the evolution operator in the complex E-plane (E being conjugate to time t): All singularities of the Fourier transform on the first Riemannian sheet of the complex E variable exist only on the real axis, by virtue of the time symmetry.

We next discuss its symmetry breaking, such as decay or dissipation phenomena, for dynamical systems with a huge number of degrees of freedom. In this case, we often meet the limit $N \rightarrow \infty$ and $V \rightarrow \infty$ (N and V being the number of degrees of freedom and the volume of the system, respectively), keeping the density $\rho = N/V$ finite. In some cases we can also use the van Hove limit [1].

The purpose of this paper is to stress that the whole temporal behavior of transition probabilities in quantum mechanical systems is reflected in the above-mentioned analytical properties.

TIME SYMMETRY AND ANALYTICAL PROPERTY

The quantum-mechanical *evolution operator* is given by

$$\hat{U}(t) = \frac{i}{2\pi\hbar} \oint_C \frac{1}{\lambda - \hat{H}} e^{-i\lambda t/\hbar} d\lambda, \quad \hat{U}(0) = 1, \tag{1}$$

where $\hat{H}$ stands for the Hamiltonian operator and C for the integration contour running clockwise around all poles (see Figure 1). In this context we know that the *resolvent* $(\lambda - \hat{H})^{-1}$ is the Fourier transform of the evolution operator.

For the sake of convenience, we first consider a finite system put in a finite box. In this case the Hamiltonian operator has a discrete spectrum ranging from $E_{\min}$ (say, 0) to ∞ and correspondingly, the above resolvent has simple poles distributed from 0 to ∞ on the real axis. See Fig.1. The time symmetry is self-evident.

In order to formulate the initial- and final-value problems, we have to examine the exponential factor $e^{-i\lambda t/\hbar}$ on the complex λ-plane:

$$|e^{-\frac{i}{\hbar}\lambda t}| = e^{+\frac{1}{\hbar}\mathrm{Im}\lambda t} \xrightarrow{|\lambda| \to \infty} 0 \begin{cases} \text{for } \mathrm{Im}\lambda < 0 \text{ and } t > 0, \\ \\ \text{for } \mathrm{Im}\lambda > 0 \text{ and } t < 0. \end{cases} \tag{2}$$

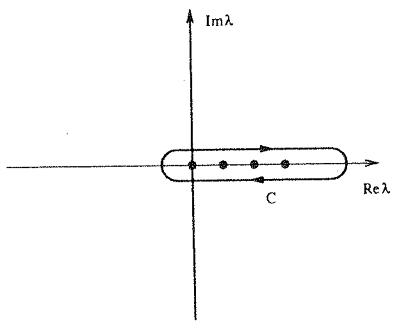

Figure 1. Contour C and singularities.

Consequently, the initial- and final-value problems can be set up by choosing the contours $C_\mp$ shown in Figure 2.

Note that

$$\lim_{R\to\infty} \int_{C_\mp} \frac{1}{\lambda - \hat{H}} e^{-i\lambda t/\hbar} d\lambda = 0, \tag{3}$$

where we take the lower and upper semicircles, $C_\mp$, corresponding to the initial- and final-value problems, and R stands for the radius of the semicircles. The time symmetry is still clear.

Here let us take the infinite N and V limit as mentioned in the Introduction, in order to deal with dynamical systems with a huge number of degrees of freedom. In this limit, the above discrete poles of the resolvent merge to make a continuous distribution of simple poles on the real axis, and a pair of (new) poles appear in the next Riemannian sheet in a symmetrical way with respect to the real axis. This kind of symmetric structure (of singularities) with respect to the real axis is also a reflection of the time symmetry. Furthermore, we have to remember that such a continuous distribution of poles is equivalent to a branch cut running from a branch point at $\lambda = 0$ to ∞.

INITIAL-VALUE PROBLEM—THREE-STEP STRUCTURE

Consider the initial-value problem, such as decay or dissipation phenomena. Taking into account the behavior (2) and the above-mentioned analytical property of the integrand, we know that the integral (1) vanishes for $t < 0$—this means that we are dealing with the initial-value problem. Because of the relation (3), we are led to a contour integral along a straight line running just above the real axis from $-\infty$ to $+\infty$. (Contrary to this, the final-value problem is specified by a vanishing evolution operator for $t > 0$ and the integration path should be taken along a straight line running just below the real axis from $-\infty$ to $+\infty$.)

Thus, we rewrite the λ-integral (1) for the initial-value problem as a sum of contributions coming from a (new) pole locating on the lower half-plane of the second Riemannian sheet, and from a contour running just above the real axis from $-\infty$ to 0 in the first Riemannian sheet, going round the branch point 0 and running just below

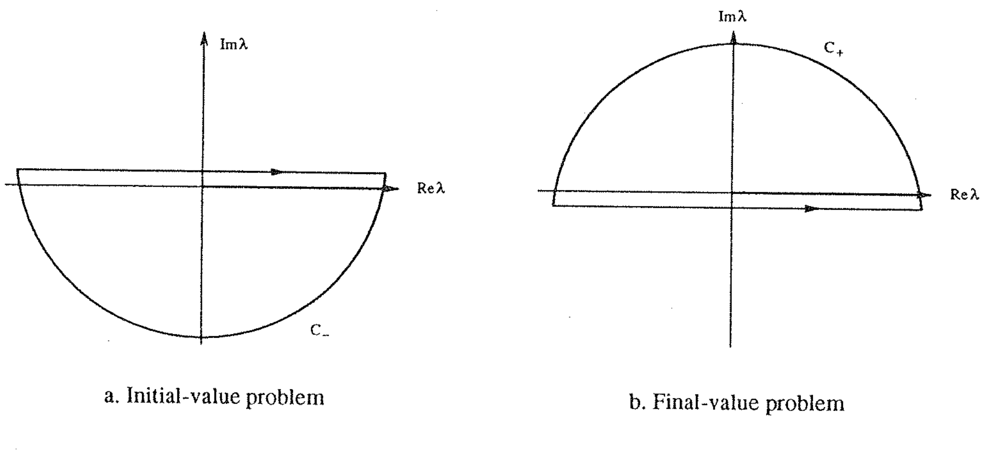

Figure 2. Contours for the initial- and final-value problems.

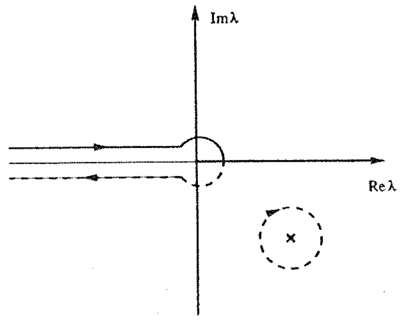

Figure 3. Decomposition of singularities for the initial-value problem.

the real axis from 0 to $-\infty$ in the second Riemannian sheet (see Figure 3). From the perturbation-theoretical point of view, we put

$$\hat{H} = \hat{H}_0 + \hat{H}', \tag{4}$$

in which the latter operator is expected to yield only small contributions in comparison with the former. In order to make our discussion clear, we naturally assume that the interaction Hamiltonian $\hat{H}'$ has no diagonal elements in the diagonal representation of $\hat{H}_0$. For details, see [2] and the Appendix, which conveniently shows the structure of the resolvent in the case of perturbation theory.

Now we can examine the whole temporal evolution of the transition probability on the basis of the above consideration. For very short times, in general, we easily derive the following formula

$$P_a \equiv |\langle a|\hat{U}(t)|a\rangle|^2 = 1 - \frac{1}{\hbar^2}(\Delta H)_a^2 t^2 + \cdots \simeq \exp[-(\Delta H)_a^2 t^2/\hbar^2], \tag{5}$$

where $(\Delta H)_a^2 \equiv \langle a|\hat{H}^2|a\rangle - \langle a|\hat{H}|a\rangle^2$ and $|a\rangle$ is chosen to be an eigenstate of $\hat{H}_0$. See [3]. This type of decay, i.e. the Gaussian decay is a sound base for the derivation of the so-called "quantum Zeno effect." See [4, 5].

In usual cases, the contribution coming from the pole in the second Riemannian sheet describes the exponential decay after the earlier stage of Gaussian decay. For longer times, after the exponential region is over, the remaining contour integral appears to describe a decay phenomenon of the power type [3, 6].

Up to the present, we have never observed the decay of the power type in usual decay processes. The contour integral cancels the pole contribution, leaving the Gaussian decay for very short times and the power decay for very long times. Remember that the Gaussian-like and power-like decays disappear in the van Hove limit [1]. In this limit, the whole process is described only in terms of the exponential decay. Even before taking this limit, when the exponential decay dominates the whole process, we can say that we are just in a dephasing process which is governed by a master equation.

Summarizing, the whole temporal behavior of the transition probability is composed of the following three steps (see also Figure 4):

$$P_a \simeq \exp[-\frac{1}{\hbar^2}(\Delta H)_a^2 t^2], \quad \text{for very short times,} \tag{6}$$

$$P_a \simeq \exp[-\Gamma_a t], \quad \text{for intermediate times,} \tag{7}$$

$$P_a \simeq \text{Const.} \times t^{-\gamma_a}, \quad \text{for very long times.} \tag{8}$$

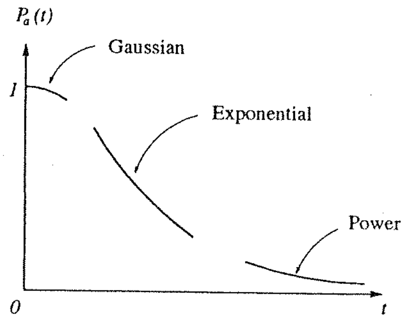

Figure 4. The three-step structure of the transition probability.

For the final-value problem, we observe a similar kind of description of decay or dissipation phenomena for negative t. This is also a reflection of the time symmetry in quantum mechanics.

QUANTUM ZENO EFFECT

Remember that at very short times, the survival probability $|\langle a|\hat{U}(t)|a\rangle|^2$ behaves quadratically

$$|\langle a|\hat{U}(t)|a\rangle|^2 = 1 - \frac{1}{\hbar^2}(\Delta\hat{H})_a^2 t^2 + \cdots \simeq \exp\left[-(\Delta\hat{H})_a^2 t^2/\hbar^2\right]. \tag{9}$$

In particular, its time derivative takes the maximum value 0 at $t=0$ and then *decreases* as t increases (within the range of validity of the above approximation)

$$0 = \frac{d}{dt}|\langle a|\hat{U}(t)|a\rangle|^2\bigg|_{t=0} > \frac{d}{dt}|\langle a|\hat{U}(t)|a\rangle|^2\bigg|_{t>0}. \tag{10}$$

This is to be contrasted with the behavior in the familiar exponential decay, in which the time derivative *increases*

$$-\Gamma_a = \frac{d}{dt}\exp[-\Gamma_a t]\bigg|_{t=0} < \frac{d}{dt}\exp[-\Gamma_a t]\bigg|_{t>0}. \tag{11}$$

From the above property (10), it is not difficult to realize that frequent interruptions of the quantum mechanical evolution of the system and the projection to the initial state at each moment of interruption drastically change its original time evolution. This phenomena is due entirely to the quantum nature of the time evolution and is called the *Quantum Zeno effect* (see References at the end of this paper), after the famous Greek philosopher Zeno and the paradox put forward by him. In fact, when the system, initially prepared in a state $|a\rangle$, is evolving quadratically at time Δt and is "observed" (either by measuring it to know if it is in the state $|a\rangle$, or by spectrally decomposing it to retain only those systems in the state $|a\rangle$), its evolution starts anew at $t = \Delta t$. If this kind of observation is repeated N times within a fixed duration $T = N\Delta t < \infty$, we can see that the more frequent the observation is, the higher its probability becomes

$$P_a^{(N)}(T) > P_a^{(N')}(T), \quad \text{for } N > N' > 1. \tag{12}$$

Frequent observations result in a drastic change in the quantum mechanical time evolution. In particular, in the ideal limit $N \to \infty$, corresponding to a "continuous observation,"

$$P_a^{(N)}(T) \longrightarrow 1 \qquad (\text{as } N \to \infty), \tag{13}$$

which implies that the system is frozen in the initial (generally unstable) state and does not decay! This paradoxical result is called the *Quantum Zeno paradox*, the notion of which was already known to von Neumann [7] and was later formulated mathematically by Misra and Sudarshan [5].

A few remarks are in order here. First, observe that this effect is entirely due to the quantum nature of the time evolution at short times: Observation of the system unavoidably affects its temporal behavior. In this context, the issue seems to have relevance to the so-called quantum measurement problem [8] and has been discussed in relation to the von Neumann projection postulate [7] since the report of a beautiful experiment performed by Itano *et al.* [9]. It is, however, now clear that their experiment is by no means a proof of the von Neumann projection postulate: The experimental results are completely explained in terms only of the dynamical evolution [10]. Second, the $N = \infty$ limit, which results in the quantum Zeno paradox, can not be meaningful, not only from an actual and technical view point but also from a physical point of view. It is shown [11] that the limit is in contradiction to the Heisenberg uncertainty relation: The limit has a meaning just as a mathematical idealization.

DEPHASING AND DISSIPATION

It would be interesting to note that the time reversal symmetry is completely lost if the system decays exponentially, while it is preserved at very short times since the probability decays quadratically in t. As has been clarified in the previous sections, the exponential decay form appears at intermediate times, i.e. after the first stage of Gaussian behavior and before the last stage of power behavior. Since the exponential decay form is representative of a purely probabilistic classical process, we can say that no quantum phase correlations are kept (i.e. we have decoherence or dephasing) in such a process, if the exponential decay dominates over the whole process. This observation may most clearly be seen in the fact that the quantum decay rate Γ_a is given by the summation, not of the amplitudes, but of the probabilities. Remember, for example, the Fermi Golden Rule

$$P_a(t) \sim e^{-\Gamma_a t}, \qquad \Gamma_a = \sum_{n;E_n=E_a} \frac{2\pi}{\hbar} \left|\langle n|\hat{H}'|a\rangle\right|^2, \tag{14}$$

which is the lowest order approximation of the perturbation theory. As is well known, we have only to replace $\hat{H}'$ with the corresponding T-matrix (i.e. $\hat{T} = \hat{H}' + \hat{H}'(\lambda - \hat{H})^{-1}\hat{H}'$), in order get the generalized formula. Taking this fact into account, we can easily understand that dephasing takes place so that no phase correlation is left in the summation. Remark that the exponential period is fully described by means of master equation, in which we only observe decoherence or dephasing.

On the other hand, we have also seen that the exponential behavior of the probability amplitude $\langle a|\hat{U}(t)|a\rangle$ is ascribed to the simple pole appearing on the second Riemannian sheet in the $N \to \infty$ limit. It is important to note that this infinite N limit plays a crucial role in this respect: Actually it transforms the discrete series of simple poles on the real E axis into a cut and, at the same time, the discrete energy levels $|n\rangle$ into continuous ones $|E\rangle$, allowing the state $|a\rangle$ to make transitions to the degenerate states $|E\rangle$ with $E = E_a$. The above decay rate should be written as

$$\Gamma_a = \int dE\rho(E)\frac{2\pi}{\hbar}\left|\langle E|\hat{H}'|a\rangle\right|^2 \delta(E - E_a), \tag{15}$$

where, for simplicity, we have used again the Fermi Golden Rule. Its generalization is obtained by replacing $\hat{H}'$ with the corresponding T-matrix, as mentioned above.

The transition from the initial Gaussian stage to the next exponential one appears quite interesting in many respects, since the latter is considered to represent a dephasing, a dissipative and also a time-symmetry breaking process. Further study of this transient region may disclose interesting interrelation between these notions.

SUMMARY

We have investigated the time symmetry of quantum systems on the basis of the analysis of the Fourier transform of the probability amplitude $\langle a|\hat{U}(t)|a\rangle$. It is shown that its t dependence is closely connected to the singularities of its Fourier transform in the complex E-plane and there are three stages of its time development: In the $N \to \infty$ limit, a simple pole appears on the second Riemannian sheet, which is responsible for the exponential decay at intermediate times, while "contributions" arising from the cut will be dominant for very short and very long times to yield Gaussian and power behaviors, respectively.

The Gaussian behavior at very short times is considered to be a simple and still general consequence of quantum theory and is shown to yield the quantum Zeno effect. On the contrary, the derivation of the exponential decay within the framework of quantum theory is by no means trivial, since it requires the derivation of a *time-asymmetric* process from the *time-symmetric* Schrödinger equation. In this respect, the $N \to \infty$ limit plays a crucial role in deriving the exponential decay form, which implies a complete loss of quantum coherence (decoherence or dephasing) and the existence of underlying dissipative process.

APPENDIX: PERTURBATIVE APPROACH TO THE RESOLVENT

Assuming that the interaction Hamiltonian $\hat{H}'$ yields a small contribution in comparison with the total Hamiltonian $\hat{H} = \hat{H}_0 + \hat{H}'$, $\hat{H}_0$ being the free Hamiltonian, we can obtain the following perturbation series of the resolvent in (1) with respect to $\hat{H}'$:

$$\begin{aligned}
\frac{1}{\lambda - \hat{H}} &= \frac{1}{\lambda - \hat{H}_0} + \frac{1}{\lambda - \hat{H}_0}\hat{H}'\frac{1}{\lambda - \hat{H}_0} \\
&\quad + \frac{1}{\lambda - \hat{H}_0}\hat{H}'\frac{1}{\lambda - \hat{H}_0}\hat{H}'\frac{1}{\lambda - \hat{H}_0} \\
&\quad + \frac{1}{\lambda - \hat{H}_0}\hat{H}'\frac{1}{\lambda - \hat{H}_0}\hat{H}'\frac{1}{\lambda - \hat{H}_0}\hat{H}'\frac{1}{\lambda - \hat{H}_0} \\
&\quad + \frac{1}{\lambda - \hat{H}_0}\hat{H}'\frac{1}{\lambda - \hat{H}_0}\hat{H}'\frac{1}{\lambda - \hat{H}_0}\hat{H}'\frac{1}{\lambda - \hat{H}_0}\hat{H}'\frac{1}{\lambda - \hat{H}_0} \\
&\quad + \cdots .
\end{aligned} \tag{16}$$

This expansion has a definite meaning under the condition

$$\langle a|\hat{H}'|a\rangle = 0 \quad \text{for} \quad \hat{H}_0|a\rangle = E_a|a\rangle, \tag{17}$$

because if this condition is not satisfied, we cannot solve the Lippman-Schwinger equation. This condition is also closely related to the *mass renormalization* in field theory and the *effective mass* in solid state physics.

Here let us introduce the so-called *random phase* approximation for a many-body system with a huge number of degrees of freedom:

$$\langle n'|\hat{H}'|n''\rangle \text{ have random phases for } n' \neq n'', \tag{18}$$

which becomes effective in the infinite N limit. If we use this approximation, we can rewrite the above perturbation series as follows:

$$\begin{aligned}
\langle a|\frac{1}{\lambda-\hat{H}}|a\rangle &= \frac{1}{\lambda-E_a} + \left(\frac{1}{\lambda-E_a}\right)^2 \left[\langle a|\hat{H}'|a\rangle + \langle a|\hat{H}'\frac{1}{\lambda-\hat{H}_0}\hat{H}'|a\rangle + \cdots\right] \\
&= \frac{1}{\lambda-E_a}\left[1 + \left(\frac{\Sigma_a(\lambda)}{\lambda-E_a}\right) + \left(\frac{\Sigma_a(\lambda)}{\lambda-E_a}\right)^2 + \cdots\right] \\
&= \frac{1}{\lambda-E_a-\Sigma_a(\lambda)}
\end{aligned} \tag{19}$$

for the diagonal element of the resolvent, and similar expressions for other transition amplitudes, where

$$\Sigma_a(\lambda) = \Sigma_a(\lambda)^{(2)} + \Sigma_a(\lambda)^{(4)} + \cdots : \tag{20}$$

$$\Sigma_a(\lambda)^{(2)} = \langle a|\hat{H}'\frac{1}{\lambda-\hat{H}_0}\hat{H}'|a\rangle = \sum_{n\neq a}\frac{|\langle a|\hat{H}'|n\rangle|^2}{\lambda-E_n}, \tag{21}$$

$$\begin{aligned}
\Sigma_a(\lambda)^{(4)} &= \langle a|\hat{H}'\frac{1}{\lambda-\hat{H}_0}\hat{H}'\frac{1}{\lambda-\hat{H}_0'}\hat{H}'\frac{1}{\lambda-\hat{H}_0'}\hat{H}'|a\rangle - \frac{1}{\lambda-E_a}\left(\Sigma_a^{(2)}\right)^2 \\
&= \sum_{n\neq a}\sum_{n'\neq a,n}\frac{|\langle a|\hat{H}'|n'\rangle|^2|\langle n'|\hat{H}'|n\rangle|^2}{(\lambda-E_n)(\lambda-E_{n'})^2}
\end{aligned} \tag{22}$$

and so on. In general, we obtain

$$\begin{aligned}
\Sigma_a(\lambda) &= \sum_{n\neq a}\frac{1}{\lambda-E_n}|\langle a|\hat{T}|n\rangle|^2 \\
&= \sum_{n\neq a}\frac{1}{\lambda-E_n}\left[|\langle a|\hat{H}'|n\rangle|^2 + \sum_{n'\neq a,n}\frac{|\langle a|\hat{H}'|n'\rangle|^2\langle n'|\hat{H}'|n\rangle|^2}{(\lambda-E_{n'})^2} + \cdots\right],
\end{aligned} \tag{23}$$

and so on. All other terms vanish by virtue of the random phase approximation. Observe that the sum rule of probability, which is self-evident in the case of the Fermi Golden Rule, is not altered.

Notice that $\hat{H}_0$ has a continuous spectrum both in the field and the many-body cases, so that the summations become integrals. This means that $\Sigma_a(\lambda)$ has a branch cut running along the real axis on the first Riemannian sheet.

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A NOVEL METHOD TO QUANTIZE SYSTEMS OF DAMPED MOTION AND IT'S APPLICATION TO NELSON'S QUANTUM MECHANICS

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ABSTRACT

First a novel method to quantize systems of damped motion is proposed in the frameworks of canonical quantization and path integral quantization. It can be afforded by considering a Lagrangian multiplied by a time-dependent function, which may represent an effective interaction with 'environment'. Next, it is shown that this method can also be represented by the framework of the Nelson's quantum mechanics.

INTRODUCTION

In physics there are numerous examples of damped motion described by phenomenological equation of motion of the form

$$\ddot{q}_0 + \eta \dot{q}_0 + \frac{\partial V}{\partial q_0} = F(t)\,. \tag{1}$$

Here η is a damping constant, $V(q_0)$ is the potential acting on q_0 and $F(t)$ is the so-called fluctuating force with the statistical properties;

$$\begin{aligned} < F(t) > &= 0\,, \\ < F(t)F(t') > &= 2\eta k T \delta(t-t')\,, \end{aligned} \tag{2}$$

where $< \cdots >$ represents the statistical average over the ensemble of identically prepared systems.

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Usually one uses the Langevin equation when one is interested in the behaviour of the system in the long time interval, for example, compared to the relaxation time of the reservoir coupled to the system. In such a case, one can ignore quantum effects and describe it by a classical theory. However, there are many other situations where quantum effects play an important role. There is a question: Given two systems, one without and one with a particular dissipative term in its classical equation of motion, what are the differences in its quantum mechanical behavior [1]? One of the approaches to this problem is that we do not attempt to quantize the dissipative system itself, but instead treat it as interacting with a complex environment and apply to the system-environment the standard quantization procedure [2, 3]. This is an interesting approach to explore, for it is possible to obtain closed equations for a dissipative quantum-mechanical system from such research. There have been tried numerous numbers of works along this line of thought, but they are far from realistic theories.

On the other hand, the employment of a time-dependent Hamiltonian would allow us to apply to the system itself the standard procedures of quantization directly. About fifty years ago Kanai proposed this type of quantization [4], however, he derived the commutation relations and Hamiltonian in an ad hoc way. In this paper we start from a time-dependent Lagrangian, set up the Hamiltonian formalism, and follow the standard procedures of quantization and also the path integral quantization. The equal time commutation relations do not change in time and this approach is free from the problems of the uncertainty principle. The application of this method to a harmonic oscillator is shown. Finally it is shown that this method can also be represented by the framework of the Nelson's quantum mechanics.

TIME-DEPENDENT LAGRANGIAN FORMALISM

Let us consider the Lagrangian,

$$L = \left(\frac{1}{2}m\dot{q}_0^2 - V(q_0)\right) f(t)\,, \tag{3}$$

where $f(t)$ is a given positive-definite and time-dependent function. Although this Lagrangian depends explicitly on time and there is no conservation law, it gives a definite action. Then, the usual action principle gives us the Euler equation

$$m\ddot{q}_0 + m\dot{q}_0\frac{\dot{f}}{f} + \frac{\partial V}{\partial q_0} = 0\,. \tag{4}$$

If we define the energy, E, of this system as the sum of kinetic energy and potential energy, it changes in time as

$$\frac{dE}{dt} = \frac{d}{dt}\left(\frac{1}{2}m\dot{q}_0^2 + V(q_0)\right) = -2\frac{\dot{f}}{f}\left(\frac{1}{2}m\dot{q}_0^2\right)\,. \tag{5}$$

Thus we have the Lagrangian which gives us a dissipative or assimilative system of q_0, if $\frac{\dot{f}}{f}$ is positive or negative. The function $f(t)$ may represent an effective interaction between the system and environment from the view point of the system-environment approach.

In order to start the canonical procedure, it is convenient to erase an energy non-conservative term in (4). For this purpose, let us introducea new variable

$$Q = q_0\sqrt{f(t)}\,. \tag{6}$$

Then, the Euler equation (4) becomes

$$m\ddot{Q} + \sqrt{f}V'\left(\frac{Q}{\sqrt{f}}\right) - m\left[\left(\frac{1}{2}\frac{d}{dt}\ln f\right)^2 + \frac{1}{2}\frac{d^2}{dt^2}\ln f\right]Q = 0\,. \tag{7}$$

Note that the linear terms of (7) are similar to the original equation($f = 1$), except that the coeficient of linear force should be changed as, $\frac{\partial V(q_0)}{\partial q_0}|_{q_0=0} \longrightarrow \frac{\partial V(q_0)}{\partial q_0}|_{q_0=0} - m\left[\left(\frac{1}{2}\frac{d}{dt}\ln f\right)^2 + \frac{1}{2}\frac{d^2}{dt^2}\ln f\right]q_0$. In the case of $\frac{\dot{f}}{f} = 2\gamma > 0$, or $f(t) = f_0 e^{2\gamma t}$, equation (4) becomes

$$m\ddot{q}_0 + 2\gamma m\dot{q}_0 + \frac{\partial V}{\partial q_0} = 0\,, \tag{8}$$

which is an equation of damped motion or Ohmic resistance. Hereafter, we study this case in detail.

CANONICAL QUANTIZATION

We can define a canonical momentum p_0 and assume the Poisson brakets:

$$p_0 \equiv \frac{\partial L}{\partial \dot{q}_0} = m\dot{q}_0 f\,, \tag{9}$$

$$\{q_0, p_0\} = 1, \quad \{q_0, q_0\} = \{p_0, p_0\} = 0\,. \tag{10}$$

Note that the canonical momentum p_0 is not $m\dot{q}_0$ in the conventional approaches, but it is multiplied by the function f. Then the Hamiltonian takes the form as

$$H = \frac{p_0^2}{2mf(t)} + V(q_0)f(t)\,, \tag{11}$$

which is explicitely time dependent. However, the Hamilton's canonical equations reproduce the Euler equation (4). The canonical framework does work even in this 'dissipative' system. Thus we can quantize this 'dissipative' system by using the canonical quantization method with the commutators,

$$[\hat{q}_0, \hat{p}_0] = i\hbar, \quad [\hat{q}_0, \hat{q}_0] = [\hat{p}_0, \hat{p}_0] = 0\,. \tag{12}$$

These commutation relations hold all the time and this approach is free from the problems of the uncertainty principle [5].

Now we will start from the Schrödinger equation in the abstract representation:

$$\left(i\hbar\frac{\partial}{\partial t} - \hat{H}\right) \mid \Psi(t)\,\rangle = 0\,. \tag{13}$$

In comparison with solving this equation directly in the q_0-representation, it is rather easy to solve it in a Q-representation:

$$\int \mid \frac{Q}{\sqrt{f(t)}}\,\rangle\,\frac{dQ}{\sqrt{f(t)}}\,\langle\,\frac{Q}{\sqrt{f(t)}} \mid = 1\,, \tag{14}$$

$$\langle\, p_0 \mid \frac{Q}{\sqrt{f(t)}}\,\rangle = \frac{1}{\sqrt{2\pi\hbar}}\exp\left(i\frac{p_0 Q}{\hbar\sqrt{f(t)}}\right)\,. \tag{15}$$

In this new representation, the Schrödinger equation is written as,

$$\int \Big\langle \frac{Q}{\sqrt{f(t)}} \Big| i\hbar\frac{\partial}{\partial t} - \hat{H} \Big| \frac{Q'}{\sqrt{f(t)}} \Big\rangle \frac{dQ'}{\sqrt{f(t)}} \Big\langle \frac{Q'}{\sqrt{f(t)}} \Big| \Psi(t) \Big\rangle = 0\,, \tag{16}$$

where the last factor is a Schrödinger wave function in the Q-representation,

$$\Big\langle \frac{Q'}{\sqrt{f(t)}} \Big| \Psi(t) \Big\rangle \equiv \Psi(Q', t)\,. \tag{17}$$

It is easy to calculate the following partial derivative of the prefactor with respect to the time variable:

$$\begin{aligned} &\Big\langle \frac{Q}{\sqrt{f(t)}} \Big| \frac{\partial}{\partial t} \Big| \frac{Q'}{\sqrt{f(t)}} \Big\rangle \frac{1}{\sqrt{f(t)}} \\ =\; & \frac{1}{2\pi\hbar} \int dp_0 e^{-i\frac{p_0 Q}{\hbar\sqrt{f}}} \frac{\partial}{\partial t}\Big(e^{i\frac{p_0 Q'}{\hbar\sqrt{f}}} \frac{1}{\sqrt{f}}\Big) \\ =\; & \frac{1}{2}\Big(\frac{d}{dt}\ln f\Big)\, \delta(Q-Q')\Big(Q'\frac{\partial}{\partial Q'}\Big)\,. \end{aligned} \tag{18}$$

Therefore the equation (16) becomes

$$\int dQ' \delta(Q-Q') \left[i\hbar\frac{\partial}{\partial t} - \hat{H}_{q_0=\frac{Q'}{\sqrt{f}}} + i\hbar\frac{1}{2}\Big(\frac{d}{dt}\ln f\Big) Q'\frac{\partial}{\partial Q'} \right] \Psi(Q',t) = 0\,, \tag{19}$$

which gives the Schrödinger equation in the Q-representation as

$$\left[i\hbar\frac{\partial}{\partial t} - \hat{H}_{\rm eff} \right] \Psi(Q,t) = 0\,, \tag{20}$$

where

$$\hat{H}_{\rm eff} = \hat{H}_{q_0=\frac{Q}{\sqrt{f}}} - i\hbar\frac{1}{2}\left(\frac{d}{dt}\ln f\right) Q\frac{\partial}{\partial Q}\,. \tag{21}$$

The effective Hamiltonian can be written as follows:

$$\begin{aligned} \hat{H}_{\rm eff} &= \hat{H}_{q_0=\frac{Q}{\sqrt{f}}} + \frac{1}{2}\left(\frac{d}{dt}\ln f\right) \hat{Q}\hat{P} \\ &= \frac{1}{2m}\left(\hat{P} + \frac{m}{2}\Big(\frac{d}{dt}\ln f\Big)\hat{Q}\right)^2 + fV\left(\frac{\hat{Q}}{\sqrt{f}}\right) \\ &\quad -\frac{m}{8}\left(\frac{d}{dt}\ln f\right)^2 \hat{Q}^2 + i\frac{\hbar}{4}\frac{d}{dt}\ln f\,, \end{aligned} \tag{22}$$

where $\hat{P}$ is the conjugate momentum of $\hat{Q}$. Note that this Hamiltonian is nonhermitian. Therefore, one may suspect that a total probability does not conserve.

Let us introduce new canonical variables:

$$\hat{q} = \hat{Q}\,, \tag{23}$$

$$\hat{p} = \hat{P} + \frac{m}{2}\frac{d}{dt}\ln f\,\hat{Q}\,. \tag{24}$$

Then $\hat{H}_{\rm eff}$ takes the following form,

$$\hat{H}_{\rm eff}(\hat{q},\hat{p}) = \frac{1}{2m}\hat{p}^2 + fV\left(\frac{\hat{q}}{\sqrt{f}}\right) - \frac{m}{8}\left(\frac{d}{dt}\ln f\right)^2 \hat{q}^2 + i\frac{\hbar}{4}\frac{d}{dt}\ln f\,. \tag{25}$$

Suppose we have a solution of (25) in a factorized form as

$$\Psi(q,t) = \exp\left(-\frac{i}{\hbar}\Big(E + i\frac{\hbar}{4t}\ln f\Big)t\right)\phi(q)\,, \tag{26}$$

Equations (20), (25) and (26) give a "stationary" Schrödinger equation

$$\hat{H}_{\rm st}(\hat{q},\hat{p})\phi(q) = E\phi(q)\,, \tag{27}$$

where

$$\hat{H}_{\rm st}(\hat{q},\hat{p}) = \frac{1}{2m}\hat{p}^2 + fV\left(\frac{\hat{q}}{\sqrt{f}}\right) - \frac{m}{8}\left(\frac{d}{dt}\ln f\right)^2 \hat{q}^2\,. \tag{28}$$

If the potential is expanded in power series of q_0 as,

$$V(q_0) = \frac{1}{2}m\omega_0^2 q_0^2 + \text{higher order terms}\,, \tag{29}$$

the effects of the function f are renormalized into the quadratic terms in the Hamiltonian as the frequency shift $\omega_0^2 \to \omega_0^2 - \left(\frac{1}{2}\frac{d}{dt}\ln f\right)^2$. When one wants to perform a perturbative calculation of some dissipative systems, this provides a firm base of the unperturbative system. Furthermore we can see that its solution has a proper normalization condition:

$$\begin{aligned} 1 &= \int \langle\, \Psi \mid \frac{Q}{\sqrt{f}}\,\rangle \frac{dQ}{\sqrt{f}} \langle\, \frac{Q}{\sqrt{f}} \mid \Psi\,\rangle \\ &= \int \frac{dq}{\sqrt{f}} e^{\frac{1}{2}\ln f}|\phi(q)|^2 \\ &= \int dq|\phi(q)|^2\,. \end{aligned} \tag{30}$$

Likewise, we get the matrix element of a physical quantity $F(\hat{q}_0,\hat{p}_0)$ as

$$\begin{aligned} \langle\, \Psi_{\rm f} \mid F(\hat{q}_0,\hat{p}_0) \mid \Psi_{\rm i}\,\rangle &= \int dq e^{\frac{i}{\hbar}(E_{\rm f}-E_{\rm i})t}\phi_{\rm f}^*(q) \\ &\qquad F\left(\frac{q}{\sqrt{f}}, \sqrt{f}(-i\hbar\frac{\partial}{\partial q} - \frac{m}{2}\frac{d\ln f}{dt}q)\right)\phi_{\rm i}(q)\,. \end{aligned} \tag{31}$$

PATH INTEGRAL QUANTIZATION

In this section we will consider the path integral quantization and look for the explicit form of the Green's function, $G(q_0(t),t;q_0(t_0),t_0)$. Suppose we split the time interval between t_0 and t into $(N+1)$ equal pieces $\Delta t = \frac{t-t_0}{N+1}$. The Green's function now becomes

$$\begin{aligned} G(q_0(t),t;q_0(t_0),t_0) &= \lim_{\Delta t\to 0} < q_0(t) = q_{N+1}|e^{-\frac{i}{\hbar}\Delta t H(t_N+\frac{\Delta t}{2})} \\ &\times e^{-\frac{i}{\hbar}\Delta t H(t_{N-1}+\frac{\Delta t}{2})}\cdots e^{-\frac{i}{\hbar}\Delta t H(t_1+\frac{\Delta t}{2})}e^{-\frac{i}{\hbar}\Delta t H(t_0+\frac{\Delta t}{2})}|q_0(t_0)>\,, \end{aligned} \tag{32}$$

where the Hamiltonian is given in (11),

$$\hat{H}(\hat{q}_0,\hat{p}_0,t)=\frac{\hat{T}}{f(t)}+f(t)\hat{V}(\hat{q}_0),\quad \hat{T}=\frac{\hat{p}_0^2}{2m}, \tag{33}$$

and the Hamiltonian is chosen at the central time in each interval. Using the formula,

$$e^{-\frac{i}{\hbar}\Delta t(\frac{\hat{T}}{f(t)}+f(t)\hat{V})}=e^{-\frac{i}{\hbar}\Delta t\frac{\hat{T}}{f(t)}}e^{-\frac{i}{hbar}\Delta t f(t)\hat{V}}+O(\frac{(\Delta t)^2}{\hbar^2}) \tag{34}$$

and substituting a complete set for each variable $q_0(t_j)=\frac{Q_j}{\sqrt{f(t_j)}}$ at $t_j=t_0+\Delta t\cdot j$,

$$\int d\left(\frac{Q_j}{\sqrt{f(t_j)}}\right)\mid\frac{Q_j}{\sqrt{f(t_j)}}\rangle\langle\frac{Q_j}{\sqrt{f(t_j)}}\mid=1,\quad (j=1,\cdots,N), \tag{35}$$

we have

$$\begin{aligned}G\left(\frac{Q}{\sqrt{f(t)}},t\ \frac{Q_0}{\sqrt{f(t_0)}},t_0\right)&=\lim_{N\to\infty}\int d(\frac{Q_1}{\sqrt{f(t_1)}})\cdots d(\frac{Q_N}{\sqrt{f(t_N)}})\\ &\times\prod_{j=0}^{N}\langle\frac{Q_{j+1}}{\sqrt{f(t_{j+1})}}\mid e^{-\frac{i}{\hbar}\Delta t\frac{\hat{T}}{f(t_j+\frac{1}{2}\Delta t)}}e^{-\frac{i}{\hbar}\Delta t f(t_j+\frac{1}{2}\Delta t)\hat{V}}\mid\frac{Q_j}{\sqrt{f(t_j)}}\rangle\,.\end{aligned} \tag{36}$$

After a usual calculation, we obtain

$$\begin{aligned}G\left(\frac{Q}{\sqrt{f(t)}},t;\frac{Q_0}{\sqrt{f(t_0)}},t_0\right)&=\lim_{N\to\infty}\int d\left(\frac{Q_1}{\sqrt{f(t_1)}}\right)\cdots d\left(\frac{Q_N}{\sqrt{f(t_N)}}\right)\\ &\times\prod_{j=0}^{N}\sqrt{\frac{mf(t_j+\frac{1}{2}\Delta t)}{2\pi i\hbar\Delta t}}\exp\left[\frac{i\Delta t}{\hbar}\sum_{j=0}^{N-1}\left[\frac{m}{2}\left\{\left(\frac{Q_{j+1}-Q_j}{\Delta t}\right)^2\right.\right.\right.\\ &+\left(\frac{d}{dt}\ln f\right)\frac{Q_{j+1}-Q_j}{\Delta t}Q_j+\frac{1}{4}\left[\left(\frac{d}{dt}\ln f\right)^2Q_j^2\right.\\ &\left.\left.\left.\left.-2\left(\frac{d^2}{dt^2}\ln f\right)(Q_{j+1}-Q_j)Q_j\right]\right\}-f(t_j)V\left(\frac{Q_j}{\sqrt{f(t_j)}}\right)\right]\right]\\ =&\lim_{N\to\infty}\int dQ_1\cdots dQ_N\left(\frac{m}{2\pi i\hbar\Delta t}\right)^{\frac{N+1}{2}}\sqrt{f(t_0)}\\ &\times\exp\left[\frac{i}{\hbar}\int dt\left\{\frac{m}{2}\left(\dot{Q}^2-\left(\frac{d}{dt}\ln f(t)\right)\dot{Q}Q\right.\right.\right.\\ &\left.\left.\left.+\frac{1}{4}\left(\frac{d}{dt}\ln f(t)\right)^2Q^2\right)-f(t)V\left(\frac{Q}{\sqrt{f(t)}}\right)-i\frac{\hbar}{4}\frac{d}{dt}\ln f(t)\right\}\right]\,.\end{aligned} \tag{37}$$

Thus, in the continuum limit, this can be written as

$$G\left(\frac{Q}{\sqrt{f(t)}},t;\frac{Q_0}{\sqrt{f(t_0)}},t_0\right)=C\int\mathcal{D}Q\,e^{\frac{i}{\hbar}\int_{t_0}^{t}L'dt}, \tag{38}$$

where L' is an effective Lagrangian,

$$L'=\frac{m}{2}\left(\dot{Q}-\left(\frac{1}{2}\frac{d}{dt}\ln f(t)\right)\cdot Q\right)^2-f(t)V\left(\frac{Q}{\sqrt{f(t)}}\right)-i\frac{\hbar}{4}\frac{d}{dt}ln\, f(t)\,. \tag{39}$$

Now, the conjugate momentum P of Q is

$$P = m\left(\dot{Q} - \left(\frac{1}{2}\frac{d}{dt}\ln f\right)\cdot Q\right) \tag{40}$$

and the corresponding effective Hamiltonian becomes,

$$\begin{aligned} H_{\rm eff} &= \frac{P^2}{2m} + \frac{1}{2}\left(\frac{d}{dt}\ln f\right)\cdot QP + f(t)V\left(\frac{Q}{\sqrt{f}}\right) + i\frac{\hbar}{4}\frac{d}{dt}\ln f \\ &= \frac{1}{2m}p^2 + fV\left(\frac{q}{\sqrt{f}}\right) - \frac{m}{8}\left(\left(\frac{d}{dt}\ln f\right)\cdot q\right)^2 + i\frac{\hbar}{4}\frac{d}{dt}\ln f\,. \end{aligned} \tag{41}$$

This is the same as (22) and in the operator form, as was seen, the order between $\hat{P}$ and $\hat{Q}$ is very important to assure the probability conservation. The Green's function is also represented in the q-representation,

$$G\left(\frac{q(t)}{\sqrt{f(t)}}, t; \frac{q(t_0)}{\sqrt{f(t_0)}}, t_0\right) = C\int \mathcal{D}q\, e^{\frac{i}{\hbar}\int_{t_0}^{t} L'dt}\,, \tag{42}$$

where

$$L' = \frac{m}{2}\left(\dot{q} - \frac{1}{2}\left(\frac{d}{dt}\ln f\right)\cdot q\right)^2 - fV\left(\frac{q}{\sqrt{f}}\right) - i\frac{\hbar}{4}\frac{d}{dt}\ln f\,. \tag{43}$$

From (43) we have the same equation of motion (7),

$$m\ddot{q} + fV'\left(\frac{q}{\sqrt{f}}\right) - m\left[\left(\frac{1}{2}\frac{d}{dt}\ln f\right)^2 + \frac{1}{2}\frac{d^2}{dt^2}\ln f\right]q = 0\,. \tag{44}$$

In order to evaluate the path integral, we employ the fluctuation expansion, splitting the paths into a classical path $q_c(t)$ plus quantum fluctuations $\xi(t)$. The fluctuation expansion makes use of the fact that the Lagrangian decomposes into the sum of a classical part and a quantum fluctuation part,

$$L' = L_c + L_q - m\frac{d}{dt}\left(\frac{1}{4}\left(\frac{d}{dt}\ln f\right)\cdot(q_c + \xi)\xi - 2\dot{q}_c\xi\right) + O(\xi^3)\,, \tag{45}$$

where

$$L_c = \frac{m}{2}\dot{q}_c^2 - fV\left(\frac{q_c}{\sqrt{f}}\right) + \frac{m}{2}\left\{\left(\frac{1}{2}\frac{d}{dt}\ln f\right)^2 + \frac{1}{2}\frac{d^2}{dt^2}\ln f\right\}q_c^2 - i\frac{\hbar}{4}\frac{d}{dt}\ln f \tag{46}$$

$$L_q = \frac{m}{2}\dot{\xi}^2 - \frac{c(t)}{2}\xi^2 \tag{47}$$

and the time dependent coefficient $c(t)$ is given by

$$c(t) = m\left\{\frac{1}{m}V''\Big|_{\frac{q_c}{\sqrt{f}}} - \left(\frac{1}{2}\frac{d}{dt}\ln f\right)^2 - \frac{1}{2}\left(\frac{d^2}{dt^2}\ln f\right)\right\}\,. \tag{48}$$

Thus the total Green's function splits into a classical and a quantum fluctuation factors:

$$G\left(\frac{q(t)}{\sqrt{f(t)}}, t; \frac{q(t_0)}{\sqrt{f(t_0)}}, t_0\right) = \exp\left[\frac{i}{\hbar}S_c(q_c, t; q_{c0}, t_0)\right]\tilde{G}(0, t; 0, t_0)\,, \tag{49}$$

where

$$S_c(q_c, t; q_{c0}, t_0) = \int dt L_c \,, \tag{50}$$

and

$$\begin{aligned} \dot{G}(0,t;0,t_0) &= \lim_{N\to\infty} \int d\xi_1 \cdots \xi_N \left(\frac{m}{2\pi i\hbar\Delta t}\right)^{\frac{N+1}{2}} \\ &\quad \times \exp\left[\frac{i}{\hbar}\sum_{j=0}^{N}\left[\frac{m}{2\Delta t}(\xi_{j+1}-\xi_j)^2 - \frac{1}{2}\Delta t\, c_j \xi_j^2\right]\right] \\ &= C\int \mathcal{D}\xi\, e^{\frac{i}{\hbar}\int_{t_0}^{t} L_q dt} \,. \end{aligned} \tag{51}$$

Suppose we have a solution satisfying the equation

$$\left[m\frac{\partial^2}{\partial t^2} + c(t)\right] F(t, t_0) = 0 \,, \tag{52}$$

with initial conditions

$$F(t_0, t_0) = 0, \qquad \frac{\partial}{\partial t} F|_{t_0} = 1 \,, \tag{53}$$

then the quantum fluctuation term is represented by

$$\dot{G} = \left(\frac{m}{2\pi i\hbar F(t, t_0)}\right)^{\frac{1}{2}} , \tag{54}$$

and the total Green's function becomes

$$G\left(\frac{q(t)}{\sqrt{f(t)}}, t; \frac{q(t_0)}{\sqrt{f(t_0)}}, t_0\right) = \left(\frac{m}{2\pi i\hbar F(t, t_0)}\right)^{\frac{1}{2}} \exp\left[\frac{i}{\hbar} S_c(q_c, t; q_{c0}, t_0)\right] . \tag{55}$$

Finally we give an example of harmonic oscillator potential,

$$V(q_0) = \frac{m}{2}\omega_0^2 q_0^2 \tag{56}$$

with the Ohmic resistance,

$$f(t) = f(t_0) e^{2\gamma(t-t_0)} \,. \tag{57}$$

In this case the coefficient c does not depend on time and

$$\begin{aligned} c &= m\left\{\omega_0^2 - \gamma^2\right\} \\ &\equiv m\omega^2 \,. \end{aligned} \tag{58}$$

Using a classical solution, we obtain the expression

$$\begin{aligned} S_c &= \exp\left\{\frac{im\omega}{2\hbar \sin \omega(t-t_0)}\right. \\ &\quad \left. \times \left[(q^2(t) + q^2(t_0)) \cos \omega(t-t_0) - 2q(t)q(t_0)\right]\right\} \left(\frac{f(t)}{f(t_0)}\right)^{1/4} . \end{aligned} \tag{59}$$

The solution of (52) is

$$F(t, t_0) = \frac{\sin \omega(t-t_0)}{\omega} \,. \tag{60}$$

Multiplying the quantum fluctuation factor by the classical amplitude, the Green's function of the Ohmic resistance linear oscillator reads

$$G\left(\frac{q(t)}{\sqrt{f(t)}},t;\frac{q(t_0)}{\sqrt{f(t_0)}},t_0\right)=\sqrt{\frac{m\omega}{2\pi i\hbar\sin\,\omega(t-t_0)}}\left(\frac{f(t)}{f(t_0)}\right)^{1/4}$$
$$\times\exp\left\{\frac{im\omega}{2\hbar\sin\,\omega(t-t_0)}\left[(q^2(t)+q^2(t_0))\cos\,\omega(t-t_0)-2q(t)q(t_0)\right]\right\}. \qquad (61)$$

In the original q_0-representation, it becomes

$$G(q_0(t),t;q_0(t_0),t_0)=\sqrt{\frac{m\omega}{2\pi i\hbar\sin\,\omega(t-t_0)}}\left(\frac{f(t)}{f(t_0)}\right)^{1/4}$$
$$\times\exp\left\{\frac{im\omega}{2\hbar\sin\,\omega(t-t_0)}\left[(f(t)q_0^2(t)+f(t_0)q_0^2)\cos\,\omega(t-t_0)\right.\right.$$
$$\left.\left.-2\sqrt{f(t)f(t_0)}q_0(t)q_0(t_0))\right]\right\}. \qquad (62)$$

When we start from an initial wave packet located at $q_0(0)=a$,

$$\Psi(q_0,0)=\left(\frac{\alpha}{\pi}\right)^{1/4}e^{-\frac{\alpha}{2}(q_0-a)^2},\quad \alpha=\frac{m\omega}{\hbar}, \qquad (63)$$

we obtain the wave function at time t,

$$\Psi(q,t)=\left(\frac{\alpha f(t)}{\pi}\right)^{1/4}\exp\left[-\frac{\alpha}{2}(q-a\cos\omega t)^2\right]$$
$$\times\exp\left[-i\frac{1}{2}\omega t-i\alpha a\sin\omega t(q-\frac{1}{2}a\cos\omega t)\right], \qquad (64)$$

where we set $f(0)=1$. The wave function is rewritten as

$$\Psi(q_0,t)=\left(\frac{\alpha f(t)}{\pi}\right)^{1/4}\exp\left[-\frac{\alpha f}{2}(q_0-\frac{a}{\sqrt{f}}\cos\omega t)^2\right]$$
$$\times\exp\left[-i\frac{1}{2}\omega t-i\alpha a\sqrt{f}\sin\omega t(q_0-\frac{1}{2\sqrt{f}}a\cos\omega t)\right] \qquad (65)$$

in the original coordinate variable.

Using (65), we have the expectation values of q_0 and p_0 as follow;

$$<q_0>=\frac{a}{\sqrt{f}}\cos\omega t, \qquad (66)$$
$$<q_0^2>=\frac{1}{f}\left(\frac{\hbar}{2m\omega}+a^2\cos^2\omega t\right), \qquad (67)$$
$$<p_0>=-m\omega a\sqrt{f}\sin\omega t, \qquad (68)$$
$$<p_0^2>=m\omega\hbar f\left(\frac{1}{2}+\frac{m\omega}{\hbar}a^2 f\ sin^2\omega t\right). \qquad (69)$$

Thus, in the case of a minimum wave packet, the mean square value of the coordinate variable,

$$<(\Delta q_0)^2>=\frac{\hbar}{2m\omega f}, \qquad (70)$$

damps off, while that of the momentum variable,

$$< (\Delta p_0)^2 >= \frac{m\omega\hbar}{2} f \,, \tag{71}$$

blows up, but the uncertainty relation remains constant,

$$\Delta q_0 \Delta p_0 = \frac{\hbar}{2} \,. \tag{72}$$

APPLICATION TO NELSON'S QUANTUM MECHANICS

It is very interesting to extend the Nelson's quantum mechanics [6] to the system with damped motion. It gives an explicte time-dependent behavior of quantum system [7–9]. We will start from the Hamiltonian (11), and it gives us Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = \left[-\frac{\hbar^2}{2mf}\nabla^2 + fV\right]\psi \,. \tag{73}$$

Note that this equation is rewritten as

$$i\tilde{\hbar}\frac{\partial\psi}{\partial t} = \left[-\frac{\tilde{\hbar}^2}{2m}\nabla^2 + V\right]\psi \,, \tag{74}$$

where $\tilde{\hbar} \equiv \frac{\hbar}{f}$, ($f > 0$). This relation suggests the following Langevin equations:

$$\begin{aligned} dx(t) &= b(t)dt + \frac{1}{\sqrt{f}}dw(t), \qquad \text{forward}, & (75) \\ d_*x(t) &= b_*(t)dt + \frac{1}{\sqrt{f}}dw_*(t), \ \text{backward}\,, & (76) \end{aligned}$$

where

$$< dw(t)dw(t) >= \frac{\hbar}{m}dt, \quad < dw_*(t)dw_*(t) >= -\frac{\hbar}{m}dt \,. \tag{77}$$

It is easily shown that Eqs.(75) and (76) give the following equivalent Fokker-Planck equations for the distribution function $P(x,t)$ of the random variable x(t):

$$\begin{aligned} \frac{\partial P}{\partial t} &= \left[-\frac{\partial}{\partial x}b + \frac{\hbar}{2mf}\frac{\partial^2}{\partial x^2}\right]P, & (78) \\ -\frac{\partial P}{\partial t} &= \left[\frac{\partial}{\partial x}b_* + \frac{\hbar}{2mf}\frac{\partial^2}{\partial x^2}\right]P\,, & (79) \end{aligned}$$

where

$$\begin{aligned} u &\equiv \frac{b-b_*}{2} = \frac{\hbar}{2mf}\frac{1}{P}\frac{\partial P}{\partial x}, & (80) \\ v &\equiv \frac{b+b_*}{2}\,. & (81) \end{aligned}$$

From (78) and (79) we have a continuity equation,

$$\frac{\partial P}{\partial t} + \frac{\partial}{\partial x}(vP) = 0 \,. \tag{82}$$

Partial derivation of (82) with respect to x gives a kinetic equation:

$$\frac{\partial}{\partial t}(fu) = -\frac{\hbar}{2m}\frac{\partial^2 v}{\partial x^2} - \frac{\partial}{\partial x}(fuv)\,. \tag{83}$$

Using Ito calculus, we have second order variation of x as

$$< d_*(fdx) > \; = \; (\dot{f}b_* + f\frac{\partial b}{\partial t} + f\frac{\partial b}{\partial x}b_* - \frac{\hbar}{2m}\frac{\partial^2 b}{\partial x^2})(dt)^2, \tag{84}$$

$$< d(fd_*x) > \; = \; (\dot{f}b + f\frac{\partial b_*}{\partial t} + f\frac{\partial b_*}{\partial x}b + \frac{\hbar}{2m}\frac{\partial^2 b_*}{\partial x^2})(dt)^2\,. \tag{85}$$

Therefore, we define an acceleration, α, by the following formula:

$$\alpha f \; \equiv \; \frac{1}{2}\frac{1}{(dt)^2}(< d_*(fdx) > + < d(fd_*x) >)\;, \tag{86}$$

$$= \; \frac{\partial}{\partial t}fv + \frac{\partial}{\partial x}f(v^2 - u^2) - \frac{\hbar}{2m}\frac{\partial^2}{\partial x^2}u\,. \tag{87}$$

Now we set up a dynamical "Newton" equation:

$$\frac{\partial}{\partial t}fv = \frac{\hbar}{2m}\frac{\partial^2 u}{\partial x^2} - \frac{\partial}{\partial x}f(v^2 - u^2) - \frac{f}{m}\frac{\partial V}{\partial x}\,. \tag{88}$$

If we introduce a complex function ψ of x and t,

$$(u + iv)f \equiv \frac{\hbar}{m}\frac{1}{\psi}\frac{\partial\psi}{\partial x}\,, \tag{89}$$

we have an equation,

$$\frac{\partial}{\partial x}(\frac{\hbar}{m}\frac{1}{\psi}\frac{\partial\psi}{\partial t}) = i\frac{\partial}{\partial x}\left[\frac{\hbar^2}{2m^2}\frac{1}{\psi}\frac{\partial^2\psi}{\partial x^2} - \frac{f}{m}V\right]\,, \tag{90}$$

from which Schrödinger equation of damped system is reproduced as

$$i\hbar\frac{\partial\psi}{\partial t}\left[-\frac{\hbar^2}{2mf}\frac{\partial^2}{\partial x^2} + fV\right]\psi\,. \tag{91}$$

CONCLUDING REMARKS

A novel method to quantize systems of damped motion is proposed in the frameworks of canonical quantization and path integral quantization. Here we start from a time-dependent Lagrangian multiplied by a given time-dependent function, which may represent an effective interaction with 'environment'. The employment of this time-dependent Hamiltonian allows us to apply to the system itself the standard procedures of quantization directly. We set up the Hamiltonian formalism, and follow the standard procedures of quantization and also the path integral quantization. The equal time commutation relations do not change in time, and this approach is free from the problems of the uncertainty principle. We presented an example of minimum wave packet under the harmonic oscillator with Ohmic resistance. Finally we showed that this method is also applicable to the Nelson's quantum mechanics. Using this framework, we can investigate the time-dependent behavior of quantum system with damping effect.

We believe that this formalism gives us a firm base to calculate the effects of damping, perturbatively. However, it works well within the region where the characteristic time of dissipation, $1/\gamma$, is much larger than that of the quantum system, for example, $1/\omega$ in a harmonic case. Furthermore, $f(t)$ is a given function, and its connection to the environment is not considered in this paper. In principle it should be derived from the investigation of dynamics of the system-environment.

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TEMPORAL BEHAVIOR OF QUANTUM SYSTEMS AND QUANTUM ZENO EFFECT

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ABSTRACT

The temporal behavior of an unstable system is analyzed quantum mechanically and compared to the exponential decay law. The general mathematical features of the quantum evolution, yielding a quadratic region at short times and a power law at long times, are briefly reviewed.

The consequences of the short-time quadratic evolution are curious: By performing many measurements in rapid succession on a quantum system, in order to check whether it is still in its initial state, one can hinder its evolution. This phenomenon is known as the quantum Zeno effect and is discussed in detail. In this respect, a specific example involving neutron spin is considered. Finally, we focus our attention on some interesting features of the evolution law.

INTRODUCTION

The decay of unstable systems closely follows an exponential law. Such a behavior has been experimentally verified with very high accuracy on many different quantum mechanical systems. Yet, the logical status of such a law is subtle and delicate, due to the unitarity of the quantum evolutions. The seminal work by Gamow [1] and by Weisskopf and Wigner [2] is based on the assumption that a pole near the real axis of the complex energy plane dominates the temporal evolution of the quantum system. This assumption leads to a spectrum of the Breit-Wigner type [3] and to the Fermi Golden Rule [4]. However, it is well known that a purely exponential decay law can be expected neither for very short [5] nor for very long [6, 7] times. The domain of validity of the exponential law is limited: the long-time power tails and the short-time quadratic

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behavior are unavoidable consequences of very general mathematical properties of the Schrödinger equation [8, 9].

The short-time behavior [10–14], in particular, turns out to be very interesting, due to the so-called quantum Zeno effect. Recent theoretical and experimental work on this subject [15, 16] has focussed on the temporal behavior of a two-level system whose Rabi oscillations, induced by an rf field, are hindered by another, "measuring" field of different frequency. This interesting experiment provoked a very lively debate. It is now almost unanimously accepted that the QZE is liable to a purely dynamical explanation [17, 18] and it has also been proposed that the above-mentioned experiment, although correctly performed, is not a direct test of the QZE and should be correctly reinterpreted [19].

In this paper, we shall outline the main features of the quantum mechanical evolution law and discuss an interesting example.

PRELIMINARIES

Let $|\psi_0\rangle$ be the wave function of a given quantum system at time $t = 0$. The evolution is governed by the unitary operator

$$U(t) = e^{-iHt}, \tag{1}$$

where H is the Hamiltonian. The "survival" or nondecay probability at time t is the square modulus of the survival amplitude

$$P(t) = |\langle\psi_0|U(t)|\psi_0\rangle|^2 = 1 - t^2/\tau_Z^2 + \cdots, \tag{2}$$

$$\tau_Z^{-1} \equiv \frac{\Delta H}{\hbar} = \frac{1}{\hbar}\left(\langle\psi_0|H^2|\psi_0\rangle - \langle\psi_0|H|\psi_0\rangle^2\right)^{1/2}. \tag{3}$$

We assumed that the series (2) converges [9, 20]. The short-time expansion is quadratic in t and therefore yields a vanishing decay rate for $t \to 0$. This quadratic behavior is in manifest contradiction with the exponential law that predicts an initial nonvanishing decay rate (the inverse of the lifetime). The quantity τ_Z will be referred to as "Zeno time," in the present paper.

Let us now look at the evolution law at long times. The exponential law turns out to be an approximation also in this case, for mathematically unavoidable reasons. Indeed, by introducing a complete orthonormal set $\{|n\rangle\}$ of eigenstates of H one easily expresses the survival probability amplitude $y(t)$ as a Fourier transform

$$y(t) = \langle\psi_0|U(t)|\psi_0\rangle = \int \omega_0(E)e^{-iEt}dE, \tag{4}$$

where

$$\omega_0(E) \equiv \sum_{E_n=E} |\langle n|\psi_0\rangle|^2 \tag{5}$$

is the energy density of the initial state and the (continuous) summation is taken over all the quantum numbers, except energy, that are necessary for the specification of a complete orthonormal set.

If one assumes, on physical grounds, that the spectrum of the total Hamiltonian is lower-bounded, so that there is a certain finite energy E_g below which the function $\omega_0(E)$ vanishes, the Paley-Wiener Theorem implies that

$$\int_{-\infty}^{\infty} \frac{|\ln|\langle\psi_0|U(t)|\psi_0\rangle||}{1+t^2}dt < \infty. \tag{6}$$

This important remark is due to Khalfin [21]. The inequality (6) implies that the survival probability cannot decay exponentially at large times. Notice that the only assumption made in the above is the existence of a finite E_g: Its very value is irrelevant and the conclusion is quite general.

QUANTUM ZENO EFFECT

The main features of the quantum Zeno effect (QZE) were well known to von Neumann [22], but were discussed in detail only later [10–14]: Roughly speaking, one exploits the quadratic behaviour (2) of the survival probability at short times in order to inhibit the quantum evolution. Let us show how this happens. Let Q be a quantum system, undergoing an evolution governed by the unitary operator (1); Q is prepared in a given initial state and N observations are performed at times $T/N, 2T/N, \cdots, (N-1)T/N, T$, in order to check whether Q is still in its initial state. After each measurement, the system is "projected" onto the quantum mechanical state representing the result of the measurement, and the evolution starts anew. The total duration of the experiment is $T = Nt$. The probability of observing the initial state at time T, after having performed the N above-mentioned measurements, reads

$$P^{(N)}(T) = [P(t)]^N = [P(T/N)]^N \simeq \left(1 - \frac{1}{\tau_Z^2}\left(\frac{T}{N}\right)^2\right)^N \overset{N \text{ large}}{\sim} e^{-T^2/\tau_Z^2 N}. \qquad (7)$$

Notice that both T and N are finite, in the above. This is the quantum Zeno *effect*: Repeated observations "slow down" the evolution and increase the probability that the system is still in the initial state at time T. In the limit of continuous observation ($N \to \infty$) one obtains the quantum Zeno *paradox*

$$P^{(N)}(T) \simeq \left(1 - \frac{1}{\tau_G^2}\left(\frac{T}{N}\right)^2\right)^N \overset{N\to\infty}{\longrightarrow} 1. \qquad (8)$$

Infinitely frequent observations halt the evolution, and completely "freeze" the initial state of the quantum system. We shall strictly distinguish the cases (7) and (8): While the first result is liable to experimental check, within some approximation, the second one is not, because it makes use of the notion of *instantaneous* measurements. Such an idea can be useful for computational purposes, but it is very misleading, in this context, because any known physical process takes place in a finite elapse of time, which can sometimes be very short on a macroscopic scale, but is usually long on a microscopic scale. This is even more true for a quantum mechanical measurement, which involves the interaction with a macroscopic device [23]. The limit (8) has been very critically analyzed [24, 25].

We shall adopt the following definition, valid a wide class of quantum systems, under very general conditions, when N is sufficiently large: we shall say that there is a quantum Zeno effect if the probability $P^{(N)}(T)$ that the system is found in its initial state after N measurements is such that

$$P^{(N)}(T) > P^{(N')}(T) \qquad \text{for } N > N'. \qquad (9)$$

There is no direct test of the QZE on truly unstable quantum systems, as the seminal proposals suggested. However, Cook's proposal [15] paved the way to experimental tests of the QZE by making use of oscillating systems. The following section deals with such a situation.

QUANTUM ZENO EFFECT WITH NEUTRON SPIN

The example to be discussed in this section was introduced in [18] and involves neutron spin. We point out a similar idea proposed by Peres [14] and the related experiment by the Innsbruck group [26]. A polarized neutron, whose speed is v and whose initial spin is up along a certain direction (say z), crosses N identical regions of longitudinal size ℓ in which there is a static magnetic field B, that provokes a rotation of the neutron spin around its axis (say x). The interaction time with each B-region is $t = \ell/v$ and the total interaction time with the magnetic field is $T = Nt$. By choosing $T = \pi/\omega$, where $\omega = \mu B/\hbar$ (μ being the neutron magnetic moment) the final neutron spin is down with probability $P_{\downarrow}^{(N)}(T) = 1$.

This elementary situation can be modified in order to obtain a quantum Zeno effect: one simply "monitors" the neutron spin at every step, by selecting and detecting the spin-down component, for instance by placing a polarized He-3 filter into the beam, which transmits the spin-up components and absorbs (i.e. measures) the spin-down component [27]. The probability that the neutron spin remains up at time T reads [18]

$$P_{\uparrow}^{(N)}(T) = \left(\cos^2 \frac{\omega t}{2}\right)^N = \left(\cos^2 \frac{\pi}{2N}\right)^N, \tag{10}$$

which obeys the definition (9) of QZE. This is an ideal result, valid as far as N is not too large and all losses are neglected and when it is assumed that the different stages act independently. In particular,

$$P_{\uparrow}^{(N)}(T) \stackrel{N\to\infty}{\longrightarrow} 1, \tag{11}$$

which signifies that the evolution of the initial (up) neutron spin is completely halted as N tends to infinity. The physical unrealizability of the $N \to \infty$ limit in the present experiment has been critically discussed in Ref. [25], where it was shown that this limit is in contradiction with the Heisenberg uncertainty principle. However, the experimental confirmation of the result (10), with N finite (say of order 10^3), appears realistic.

This simple experiment, although very instructive, makes use of an oscillating system (the neutron spin would eventually go back to its initial state, if left to follow its evolution under the action of the magnetic field). The idea is therefore at variance with the original proposals, based on truly unstable systems [10–12]. For this reason, alternative schemes were recently proposed [28, 29]. However, it is instructive to look at the physical characteristics of the evolution of an unstable quantum system, in order to focus on its distinctive features.

TEMPORAL EVOLUTION OF AN UNSTABLE QUANTUM STATE

When one considers the temporal evolution of unstable quantum mechanical systems, it is unavoidable to give a thorough description in terms of quantum field theory. The main source of difficulty, when one endeavours to compute the evolution law, is the validity of perturbative expansions. For example, in the derivation (2), one assumes that all moments of H in the state $|\psi_0\rangle$ are finite and (implicitly) that $|\psi_0\rangle$ is normalizable and belongs to the domain of definition of H [20]. If the volume of the box containing the system is not finite, the spectrum of the Hamiltonian is continuous and the Zeno time turns out to be inversely proportional to some power of a frequency cutoff Λ: $\tau_Z \propto 1/\Lambda^\alpha$. This is a very general property, due to the singular nature of the product of local observables when computed at short distances [30].

A standard method of analysis employs the Laplace transform of the survival amplitude $y(t) = \langle\psi_0|e^{-iHt}|\psi_0\rangle$

$$\widetilde{y}(s) = \int_0^\infty dt\, e^{-st} y(t) = \langle\psi_0|\frac{1}{s+iH}|\psi_0\rangle. \tag{12}$$

By solving for $\widetilde{y}(s)$, for instance by means of a perturbative expansion, and inverting the transform, one obtains

$$y(t) = \frac{1}{2\pi i}\int_{\mathrm{B}} ds\, e^{st}\widetilde{y}(s) \tag{13}$$

where B is the Bromwich path, i.e. a vertical line at the right of all the singularities of $\widetilde{y}(s)$.

In general, the denominator in $\widetilde{y}(s)$ has a branch cut extending from a finite value of s (say 0 for simplicity) to $-i\infty$, and no singularities on the first Riemannian sheet (physical sheet). This is a very general feature [9], closely related to the time-symmetry invariance [31].

On the other hand, due to the discontinuity across the cut, the function $\widetilde{y}(s)$ has a *simple* pole on the second Riemannian sheet. This remark is due to Araki et al. and Schwinger [32]. By writing $s_{\mathrm{pole}} \equiv -i\Delta E - \gamma/2$, and by deforming the original Bromwich path into a new contour $C = C_1 + C_2$, composed of a small circle C_1 turning anticlockwise around the simple pole s_{pole} on the second Riemannian sheet and a path C_2 starting from $-\infty$ on the second sheet, turning around the branch point $s = 0$ and extending back to $-\infty$ on the first sheet, one can write

$$y(t) = y_{\mathrm{pole}}(t) + y_{\mathrm{cut}}(t), \tag{14}$$

where

$$y_{\mathrm{pole}}(t) = \mathcal{Z}e^{-\frac{\gamma}{2}t}e^{-i\phi(t)}, \tag{15}$$

$\mathcal{Z}$ being a positive constant and $\phi(t)$ a real phase, whose asymptotic behavior can be determined. This is nothing but the Fermi "Golden Rule," yielding the lifetime $\tau_{\mathrm{E}} = \gamma^{-1}$: It is well known [9, 31] that the exponential law can just be obtained by neglecting the contribution arising from the branch cut.

The branch cut yields all the corrections to the exponential law, both at short and long times, in line with the analysis of Section 2. We have already discussed the situation at times much shorter than the "Zeno" time (3). Let us therefore look at long times (a good time scale is the lifetime: $t \gg \gamma^{-1} = \tau_{\mathrm{E}}$). One can show that, under general conditions,

$$y_{\mathrm{cut}}(t) \sim \mathcal{C}t^{-\alpha}, \tag{16}$$

where $\mathcal{C}$ is a constant [which can be made real by adjusting the total phase of $y(t)$] and α a positive constant that can be explicitly calculated in some interesting cases. For instance, $\alpha = 3/2$ [33] for the Lee model [34] and $\alpha = 2$ [35, 36] for atomic spontaneous emissions in the rotating wave approximation.

Summarizing, the expressions of the survival probability at short and long times are

$$P(t) \sim 1 - \frac{t^2}{\tau_{\mathrm{Z}}^2} \qquad (t \ll \tau_{\mathrm{Z}}), \tag{17}$$

$$P(t) \sim \mathcal{Z}^2 e^{-\gamma t} + \mathcal{C}^2 t^{-2\alpha} + 2\mathcal{C}\mathcal{Z}t^{-\alpha}e^{-\frac{\gamma}{2}t}\cos[\phi(t)] \quad (t \gg \gamma^{-1}). \tag{18}$$

In general, one expects a large time domain in which the system behaves exponentially with very high accuracy. However, one observes the presence of oscillations, as a simple reflection of an *interference effect* between the pole and the cut contributions to the survival amplitude (14). This conclusion, although never explicitly stated before, is implicit in the very expression (14).

CONCLUDING REMARKS

We have discussed the deviations from exponential in the temporal behavior of quantum systems. After analyzing an example involving an oscillating system (a neutron spin), whose Poincaré recurrence time is finite, we have considered, in the previous section, the evolution of an unstable quantum system.

We saw that the survival probability shows, in general, damped oscillations around the exponential, as a reflection of a sort of interference effect between the pole and the cut contribution to the survival amplitude (14). However, the *quantitative* features of this interference effect require additional investigation, in particular at the numerical level.

An interesting problem is to understand whether the initial quadratic behavior (17) is experimentally observable for a truly unstable system of the type considered in the previous section. This is an experimentally challenging and difficult task, that raises subtle theoretical and experimental questions about the problem of state preparation. In particular, one has to face the limitations arising from the time-energy uncertainty relation and the delicate issue of defining the initial moment of excitation.

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Reflection and transmission in a neutron-spin test of the quantum Zeno effect

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The dynamics of a quantum system undergoing frequent "measurements," leading to the so-called quantum Zeno effect, is examined on the basis of a neutron-spin experiment recently proposed for its demonstration. Unlike in all previous studies, the spatial degrees of freedom of the neutron are duly taken into account. Their inclusion in the analysis is important for two reasons: first, neutron-reflection effects are shown to be very important; second, the evolution may turn out to be totally different from the ideal case. Our results can be interpreted in terms of a rigorous theorem due to Misra and Sudarshan: indeed we clarify that, in contrast with a widespread belief, a quantum Zeno effect does not halt the evolution of a quantum system; it rather modifies it, by forcing the system to remain in a certain subspace, defined by the very measurement performed.
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I. INTRODUCTION

A quantum system, prepared in a state that does not belong to an eigenvalue of the total Hamiltonian, starts to evolve quadratically in time [1,2]. This characteristic behavior leads to the so-called quantum Zeno phenomenon, namely the possibility of slowing down the temporal evolution (eventually hindering transitions to states different from the initial one) [3].

The original proposals that aimed at verifying this effect involved unstable systems and were not amenable to experimental testing [4]. However, the remarkable idea [5] to use a two-level system motivated an interesting experimental test [6], revitalizing a debate on the physical meaning of this phenomenon [7,8]. There seem to be a certain consensus, nowadays, that the quantum Zeno effect (QZE) can be given a dynamical explanation, involving only an explicit Hamiltonian dynamics.

It is worth emphasizing that the discussion of the past few years mostly stemmed from experimental considerations, related to the *practical* possibility of performing experimental tests. Some examples are the interesting issue of "interaction-free" measurements [9] and the neutron-spin tests of the QZE [8,10]. In practical cases, one cannot neglect the presence of losses and imperfections, which obviously conspire against an almost-ideal experimental realization, more so when the total number of "measurements" increases above certain theoretical limits.

The aim of the present paper is to investigate an interesting (and often overlooked) feature of what we might call the quantum Zeno dynamics. We shall see that a series of "measurements" (von Neumann's projections [11]) does not necessarily hinder the evolution of the quantum system. On the contrary, the system can evolve away from its initial state, provided it remains in the subspace defined by the "measurement" itself. This interesting feature is readily understandable in terms of rigorous theorems [2], but it seems to us that it is worth clarifying it by analyzing interesting physical examples. We shall therefore focus our attention on an experiment involving neutron spin [8] and shall see that in fact this enables us to accomplish two goals: not only will the state of the neutron undergoing QZE *change*, but it will do so in a way that clarifies why reflection effects may play a substantial role in the experiment analyzed.

In the neutron-spin example to be considered, the evolution of the spin state is hindered when a series of spectral decompositions (in Wigner's sense [12]) is performed on the spin state. No "observation" of the spin states, and therefore no projection in the manner of von Neumann, is required, as far as the different branch waves of the wave function cannot interfere after the spectral decomposition. Needless to say, the analysis that follows could be performed in terms of a Hamiltonian dynamics, without making use of projection operators. However, we shall use in this paper the von Neumann technique, which will be found convenient because it sheds light on some remarkable aspects of the Zeno phenomenon and helps to pin down the physical implications of some mathematical hypotheses with relatively fewer efforts.

The paper is organized as follows. We briefly review, in the next section, the seminal theorem for the short-time dynamics of quantum systems, proved by Misra and Sudarshan [2]. Its application to the neutron-spin case is discussed in Sec. III. In Secs. IV and V, unlike in previous papers [8,10], we shall incorporate the spatial (one-dimensional, for simplicity) degrees of freedom of the neutron and represent them by an additional quantum number that labels, roughly speaking, the direction of motion of the wave packet. A more realistic analysis is presented in Sec. VI. Finally, Sec. VII is devoted to a discussion. Some additional aspects of our analysis are clarified in the Appendix.

II. MISRA AND SUDARSHAN'S THEOREM

Consider a quantum system Q, whose states belong to the Hilbert space $\mathcal{H}$ and whose evolution is described by the unitary operator $U(t)=\exp(-iHt)$, where H is a semibounded Hamiltonian. Let E be a projection operator and $E\mathcal{H}E=\mathcal{H}_E$ the subspace spanned by its eigenstates. The initial density matrix ρ_0 of system Q is taken to belong to $\mathcal{H}_E$. If Q is let to follow its "undisturbed" evolution, under the action of the Hamiltonian H (i.e., no measurements are performed in order to get informations about its quantum state), the final state at time T reads

$$\rho(T)=U(T)\rho_0 U^\dagger(T) \tag{1}$$

and the probability that the system is still in $\mathcal{H}_E$ at time T is

$$P(T)=\mathrm{Tr}[U(T)\rho_0 U^\dagger(T)E]. \tag{2}$$

We call this a "survival probability": it is in general smaller than 1, since the Hamiltonian H induces transitions out of $\mathcal{H}_E$. We shall say that the quantum system has "survived" if it is found to be in $\mathcal{H}_E$ by means of a suitable measurement process [13].

Assume that we perform a measurement at time t, in order to check whether Q has survived. Such a measurement is formally represented by the projection operator E. By definition,

$$\rho_0=E\rho_0 E, \quad \mathrm{Tr}[\rho_0 E]=1. \tag{3}$$

After the measurement, the state of Q changes into

$$\rho_0\to\rho(t)=EU(t)\rho_0 U^\dagger(t)E, \tag{4}$$

with

$$P(t)=\mathrm{Tr}[U(t)\rho_0 U^\dagger(t)E]. \tag{5}$$

This is the probability that the system has survived. [There is, of course, a probability $1-P$ that the system has not survived (i.e., it has made a transition outside $\mathcal{H}_E$) and its state has changed into $\rho'(t)=(1-E)U(t)\rho_0 U^\dagger(t)(1-E)$. Henceforth we concentrate our attention on the measurement outcome (4) and (5).] The above is the standard Copenhagen interpretation: The measurement is considered to be instantaneous. The "quantum Zeno paradox" [2] is the following. We prepare Q in the initial state ρ_0 at time 0 and perform a series of E observations at times $t_k=kT/N(k=1,\ldots,N)$. The state of Q after the above-mentioned N measurements reads

$$\rho^{(N)}(T)=V_N(T)\rho_0 V_N^\dagger(T), \quad V_N(T)\equiv[EU(T/N)E]^N, \tag{6}$$

and the probability to find the system in $\mathcal{H}_E$ ("survival probability") is given by

$$P^{(N)}(T)=\mathrm{Tr}[V_N(T)\rho_0 V_N^\dagger(T)]. \tag{7}$$

Equations (6) and (7) display the "quantum Zeno effect": repeated observations in succession modify the dynamics of the quantum system; under general conditions, if N is sufficiently large, all transitions outside $\mathcal{H}_E$ are inhibited.

In order to consider the $N\to\infty$ limit ("continuous observation"), one needs some mathematical requirements: define

$$\mathcal{V}(T)\equiv\lim_{N\to\infty}V_N(T), \tag{8}$$

provided the above limit exists in the strong sense. The final state of Q is then

$$\tilde{\rho}(T)=\mathcal{V}(T)\rho_0\mathcal{V}^\dagger(T) \tag{9}$$

and the probability to find the system in $\mathcal{H}_E$ is

$$\mathcal{P}(T)\equiv\lim_{N\to\infty}P^{(N)}(T)=\mathrm{Tr}[\mathcal{V}(T)\rho_0\mathcal{V}^\dagger(T)]. \tag{10}$$

One should carefully notice that nothing is said about the final state $\tilde{\rho}(T)$, which depends on the characteristics of the model investigated and on the *very measurement performed* (i.e., on the projection operator E, which enters in the definition of V_N). Misra and Sudarshan assumed, on physical grounds, the strong continuity of $\mathcal{V}(t)$,

$$\lim_{t\to 0^+}\mathcal{V}(t)=E, \tag{11}$$

and proved that under general conditions the operators $\mathcal{V}(T)$ exist for all real T and form a semigroup labeled by the time parameter T. Moreover, $\mathcal{V}^\dagger(T)=\mathcal{V}(-T)$, so that $\mathcal{V}^\dagger(T)\mathcal{V}(T)=E$. This implies, by Eq. (3), that

$$\mathcal{P}(T)=\mathrm{Tr}[\rho_0\mathcal{V}^\dagger(T)\mathcal{V}(T)]=\mathrm{Tr}[\rho_0 E]=1. \tag{12}$$

If the particle is "continuously" observed, in order to check whether it has survived inside $\mathcal{H}_E$, it will never make a transition to $\mathcal{H}-\mathcal{H}_E$. This is the "quantum Zeno paradox."

An important remark is now in order: the theorem just summarized *does not* state that the system *remains* in its initial state after the series of very frequent measurements. Rather, the system is left in the subspace $\mathcal{H}_E$, instead of evolving "naturally" in the total Hilbert space $\mathcal{H}$. This subtle point, implied by Eqs. (9)–(12), is often not duly stressed in the literature.

Notice also the conceptual gap between Eqs. (7) and (10): To perform an experiment with N finite is only a practical problem, from the physical point of view. On the other hand, the $N\to\infty$ case is physically unattainable, and is rather to be regarded as a mathematical limit (although a very interesting one). In this paper, we shall not be concerned with this problem (thoroughly investigated in [10]) and shall consider the $N\to\infty$ limit for simplicity. This will make the analysis more transparent.

III. QUANTUM ZENO EFFECT WITH NEUTRON SPIN

The example we consider is a neutron spin in a magnetic field [8]. (A photon analog was first outlined by Peres [14].)

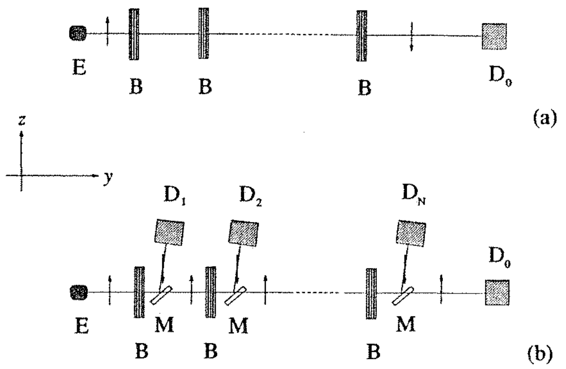

FIG. 1. (a) Evolution of the neutron spin under the action of a magnetic field. An emitter sends a spin-up neutron through several regions where a magnetic field B is present. The detector D_0 detects a spin-down neutron: No Zeno effect occurs. (b) Quantum Zeno effect: the neutron spin is "monitored" at every step, by selecting and detecting the spin-down component. D_0 detects a spin-up neutron.

We shall consider two different experiments: Refer to Figs. 1(a) and 1(b). In the case schematized in Fig. 1(a), the neutron interacts with several identical regions in which there is a static magnetic field B, oriented along the x direction. We neglect here any losses and assume that the interaction is given by the Hamiltonian

$$H=\mu B\sigma_1, \tag{13}$$

μ being the (modulus of the) neutron magnetic moment, and σ_i $(i=1,2,3)$ the Pauli matrices. We denote the spin states of the neutron along the z axis by $|\uparrow\rangle$ and $|\downarrow\rangle$.

Let the initial neutron state be $\rho_0=\rho_{\uparrow\uparrow}\equiv|\uparrow\rangle\langle\uparrow|$. The interaction with the magnetic field provokes a rotation of the spin around the x direction. After crossing the whole setup, the final density matrix reads

$$\rho(T)\equiv e^{-iHT}\rho_0 e^{iHT}=\cos^2\frac{\omega T}{2}\rho_{\uparrow\uparrow}+\sin^2\frac{\omega T}{2}\rho_{\downarrow\downarrow} - \frac{i}{2}\sin\omega T(\rho_{\uparrow\downarrow}-\rho_{\downarrow\uparrow}), \tag{14}$$

where $\omega=2\mu B$ and T is the total time spent in the B field. Notice that the free evolution is neglected (and so are reflection effects, wave-packet spreading, etc.). If T is chosen so as to satisfy the "matching" condition $\cos\omega T/2=0$, we obtain

$$\rho(T)=\rho_{\downarrow\downarrow} \quad \left(T=(2m+1)\frac{\pi}{\omega}, \quad m\in\mathbb{N}\right), \tag{15}$$

so that the probability that the neutron spin is down at time T is

$$P_\downarrow(T)=1 \quad \left(T=(2m+1)\frac{\pi}{\omega}, \quad m\in\mathbb{N}\right). \tag{16}$$

The above two equations correspond to Eqs. (1) and (2). In our example, H is such that if the system is initially prepared in the up state, it will evolve to the down state after time T. Notice that, within our approximations, the experimental setup described in Fig. 1(a) is equivalent to the situation where a magnetic field B is contained in a single region of space.

Let us now modify the experiment just described by inserting at every step a device able to select and detect one component [say the down ($\downarrow$) one] of the neutron spin. This can be accomplished by a magnetic mirror M and a detector D. The former acts as a "decomposer," by splitting a neutron wave with indefinite spin (a superposed state of up and down spins) into two branch waves each of which is in a definite spin state (up *or* down) along the z axis. The down state is then forwarded to a detector, as shown in Fig. 1(b). The magnetic mirror yields a spectral decomposition [12] with respect to the spin states, and can be compared to the inhomogeneous magnetic field in a typical Stern-Gerlach experiment.

We choose the same initial state for Q as in the previous experiment [Fig. 1(a)]. The action of $M+D$ is represented by the operator $E\equiv\rho_{\uparrow\uparrow}$ [remember that we follow the evolution along the horizontal direction, i.e., the direction the spin-up neutron travels, in Fig. 1(b)], so that if the process is repeated N times, as in Fig. 1(b), we obtain

$$\rho^{(N)}(T)=V_N(T)\rho_0V_N^\dagger(T)=\left(\cos^2\frac{\omega t}{2}\right)^N\rho_{\uparrow\uparrow} =\left(\cos^2\frac{\pi}{2N}\right)^N\rho_{\uparrow\uparrow}, \tag{17}$$

where the "matching" condition for $T=Nt$ [see Eq. (15)] has been required again. The probability that the neutron spin is up at time T, if N observations have been made at time intervals t $(Nt=T)$, is

$$P_\uparrow^{(N)}(T)=\left(\cos^2\frac{\pi}{2N}\right)^N. \tag{18}$$

This discloses the occurrence of a QZE: Indeed, $P_\uparrow^{(N)}(T)>P_\uparrow^{(N-1)}(T)$ for $N\geq 2$, so that the evolution is "slowed down" as N increases. Moreover, in the limit of infinitely many observations,

$$\rho^{(N)}(T)\overset{N\to\infty}{\longrightarrow}\tilde{\rho}(T)=\rho_{\uparrow\uparrow} \tag{19}$$

and

$$\mathcal{P}_\uparrow(T)\equiv\lim_{N\to\infty}P_\uparrow^{(N)}(T)=1. \tag{20}$$

Frequent observations "freeze" the neutron spin in its initial state, by inhibiting $(N\geq 2)$ and eventually hindering $(N\to\infty)$ transitions to other states. Notice the difference from Eqs. (15) and (16): The situation is completely reversed.

IV. SPATIAL DEGREES OF FREEDOM

In the analysis of the preceding section only the spin degrees of freedom were taken into account. No losses were considered, even though their importance was already mentioned in [8,10]. In spite of such a simplification, the model yields physical insight into the Zeno phenomenon, and has the nice advantage of being solvable.

We shall now consider a more detailed description. The practical realizability of this experiment has already been discussed, with particular attention to the $N\rightarrow\infty$ limit and various possible losses [10]. One source of losses is the occurrence of reflections at the boundaries of the interaction region and/or at the spectral decomposition step. A careful estimate of such effects would require a dynamical analysis of the motion of the neutron wave packet as it crosses the whole interaction region (magnetic-field regions followed by field-free regions containing each a magnetic mirror M that performs the "measurement"). However, it is not an easy task to include the spatial degrees of freedom of the neutron in the analysis; instead, we shall adopt a simplified description of the system, which preserves most of the essential features and for which an explicit solution can still be obtained. It turns out that the inclusion of the spatial degrees of freedom in the evolution of the spin state can result in completely different situations from the ideal case, which in turn clarifies the importance of losses in actual experiments and, at the same time, sheds new light on the Zeno phenomenon itself.

Let us now try to incorporate the other degrees of freedom of the neutron state in our description. Let our state space be the four-dimensional Hilbert space $\mathcal{H}_p\otimes\mathcal{H}_s$, where $\mathcal{H}_p=\{|R\rangle,|L\rangle\}$ and $\mathcal{H}_s=\{|\uparrow\rangle,|\downarrow\rangle\}$ are two-dimensional Hilbert spaces, with R (L) representing a particle traveling towards the right (left) direction along the y axis, and $\uparrow$ ($\downarrow$) representing spin up (down) along the z axis. We shall set, in the respective Hilbert spaces,

$$|R\rangle=\begin{pmatrix}1\\0\end{pmatrix},\quad |L\rangle=\begin{pmatrix}0\\1\end{pmatrix};\quad |\uparrow\rangle=\begin{pmatrix}1\\0\end{pmatrix},\quad |\downarrow\rangle=\begin{pmatrix}0\\1\end{pmatrix}, \tag{21}$$

so that, for example, the state $|R\downarrow\rangle$ represents a spin-down particle traveling towards the right direction ($+y$). Also, for the sake of simplicity, we shall work with vectors, rather than density matrices (the extension is straightforward).

In this extended Hilbert space, the first Pauli matrix σ_1 acts only on $\mathcal{H}_s$ as a spin flipper, $\sigma_1|\uparrow\rangle=|\downarrow\rangle$ and $\sigma_1|\downarrow\rangle=|\uparrow\rangle$, while another first Pauli matrix τ_1 acts only on $\mathcal{H}_p$ as a direction-reversal operator, $\tau_1|R\rangle=|L\rangle$ and $\tau_1|L\rangle=|R\rangle$. To investigate the effects of reflection, we assume that the interaction is described by the Hamiltonian

$$H=g(1+\alpha\tau_1)(1+\beta\sigma_1), \tag{22}$$

where g, α, and β are real constants. By varying these parameters and the total interaction time T, the above Hamiltonian can describe various situations in which a neutron, impinging on a B field applied along x axis, undergoes transmission/reflection and/or spin-flip effects.

It is worth pointing out that the above Hamiltonian incorporates the spatial degrees of freedom in an abstract way: Only the one-dimensional motion of the neutron, represented by L and R, has been taken into account and all other effects (such as, for instance, the spread of the wave packet) are neglected. This amounts to considering a trivial free Hamiltonian, which can be dropped out from the outset. This may seem too drastic an approximation; however, it is not as rough as one may imagine. Let us consider, for example, a realization of the quantum Zeno effect experiment described in the preceding section, at present in progress at the pulsed ISIS neutron spallation source. Neutrons are trapped between perfect crystal blades and pass on each of their 2000 trajectories through a flipper device, which cause an adjustable spin rotation. Flipped neutrons immediately leave the storage system where they can be easily detected (see, e.g., [15]). In such a case, the neutron burst injected into the 1-m-long perfect crystal storage system has a momentum resolution [16]

$$\frac{\delta k}{k_0}\simeq 10^{-5}. \tag{23}$$

The burst spreads according to the classical law

$$(\Delta t)^2=(\Delta t_p)^2+\left(\frac{\delta\lambda}{\lambda}t_0\right)^2, \tag{24}$$

where $\Delta t_p\simeq 140$ μm is a typical value for the neutron burst (or alternatively can be theoretically considered as the opening time of a chopper), $\lambda=2\pi/k$, and t_0 is the time of flight in the crystal storage system. For one traverse between the crystal plates $t_0\simeq 1$ ms, while for 2000 traverses $t_0\simeq 2$ s, so that Eq. (24) yields $\Delta t\simeq 170$ μs. We clearly see that, under these conditions, the additional spread of the burst over the total distance traveled in the storage crystal is negligible. In this case, the Hamiltonian (22) describes the relevant physics with good approximation.

Since the spin flipper σ_1 and the direction-reversal operator τ_1 commute with each other and with the Hamiltonian (22), the energy levels of the system governed by this Hamiltonian are obviously $E_{\tau\sigma}\equiv g(1+\tau\alpha)(1+\sigma\beta)$ with $\tau,\sigma=\pm$. Moreover, the evolution of the system has the following factorized structure:

$$e^{-iHT}=e^{-igT}e^{-i\alpha gT\tau_1}e^{-i\beta gT\sigma_1}e^{-i\alpha\beta gT\tau_1\sigma_1}. \tag{25}$$

If a neutron is initially prepared in state $|R\uparrow\rangle$, the evolution operator is explicitly expressed as

$$e^{-iHT}=t_\uparrow+r_\uparrow\tau_1+t_\downarrow\sigma_1+r_\downarrow\tau_1\sigma_1, \tag{26}$$

where $t_\uparrow$, $t_\downarrow$, $r_\uparrow$, and $r_\downarrow$ are the transmission/reflection coefficients of a neutron, whose spin is flipped/not flipped after interacting with a constant magnetic field B, applied along the x direction in a finite region of space (square potential, stationary state problem). See Fig. 2. These coefficients are connected with the energy levels by the following relation:

$$\begin{pmatrix}t_\uparrow & t_\downarrow\\ r_\uparrow & r_\downarrow\end{pmatrix}=\frac{1}{4}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}e^{-iE_{++}T} & e^{-iE_{+-}T}\\ e^{-iE_{-+}T} & e^{-iE_{--}T}\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}. \tag{27}$$

By specifying the values of the parameters g,α,β and the total interaction time T, one univocally determines $t_\uparrow$, $t_\downarrow$, $r_\uparrow$, and $r_\downarrow$. Direct physical meaning can therefore be attributed to the constants g, α, and β in Eq. (22) by comparison with the transmission/reflection coefficients. For example, in order to mimic a realistic experimental setup with given values of $t_{\uparrow\downarrow}$, $r_{\uparrow\downarrow}$, it is enough to obtain the values of g, α, and β from Eq. (27) and insert them into the Hamiltonian

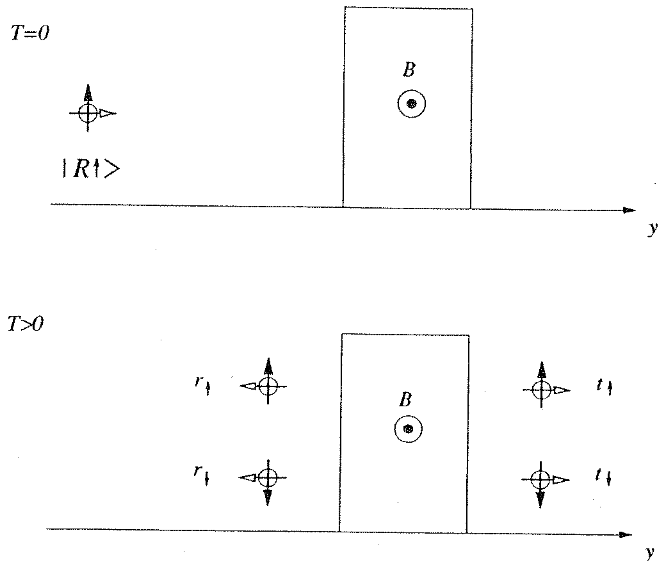

FIG. 2. Transmission and reflection coefficients for a neutron initially prepared in the $|R\uparrow\rangle$ state.

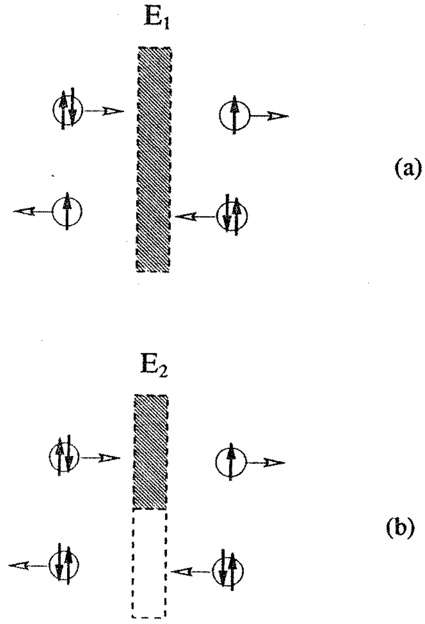

FIG. 3. (a) Direction-insensitive spin measurement. (b) Direction-sensitive spin measurement.

(22). The model could in principle be further improved by making the constant g energy-dependent. We will consider a more realistic Hamiltonian in Sec. VI.

V. IDEAL CASE OF COMPLETE TRANSMISSION

In the following discussions we always assume that our initial state is $|R\uparrow\rangle$, i.e., a right-going spin-up neutron, and consider, for definiteness, the case of total transmission with spin flipped, i.e., $|t_\downarrow|^2=1$, *when no measurements are performed.* Of course, this has to be considered as an idealized situation, since a spin rotation can only take place when there is an interaction potential (proportional to the intensity of the magnetic field) which necessarily produces reflection effects (with the only exception of plane waves). Stated differently, when the spatial degrees of freedom are taken into account in the scattering problem off a spin-flipping potential, complete transmission is impossible to achieve: There are always reflected waves. Our model Hamiltonian (22) must therefore be regarded as a simple caricature of the physical system we are analyzing. Wave-packet effects will be discussed in Sec. VI.

To obtain a total transmission with spin flipped, the evolution operator should have the form $e^{-iHT}\propto\sigma_1$, which is equivalent to either

$$e^{-i\alpha gT\tau_1}\propto\tau_1,\quad e^{-i\beta gT\sigma_1}\propto 1,\quad e^{-i\alpha\beta gT\tau_1\sigma_1}\propto\tau_1\sigma_1, \tag{28}$$

or

$$e^{-i\alpha gT\tau_1}\propto 1,\quad e^{-i\beta gT\sigma_1}\propto\sigma_1,\quad e^{-i\alpha\beta gT\tau_1\sigma_1}\propto 1. \tag{29}$$

That is,

$$\text{case (i):}\quad \cos\alpha gT=\sin\beta gT=\cos\alpha\beta gT=0, \tag{30}$$

or

$$\text{case (ii):}\quad \sin\alpha gT=\cos\beta gT=\sin\alpha\beta gT=0. \tag{31}$$

(All other cases, such as total reflection with/without spin-flip can be analyzed in a similar way.) In both cases, the evolution is readily computed:

$$e^{-iHT}|R\uparrow\rangle=(\text{phase factor})\times|R\downarrow\rangle. \tag{32}$$

The boundary conditions are such that the neutron is transmitted and its spin flipped with unit probability. For the experimental realization, see [17]. This is the situation outlined in Fig. 1(a).

We shall now focus on some interesting cases, which illustrate some definite aspects of the QZE. Let us see, in particular, how the evolution of the quantum state of the neutron is modified by choosing different projectors (corresponding to different "measurements").

A. Direction-insensitive spin measurement

We perform now a series of measurements, in order to check whether the neutron spin is up. Let us call this type of measurement a "direction-insensitive spin measurement," for reasons that will become clear later. Refer to Fig. 3(a). The projection operator corresponding to this measurement is

$$E_1=1-|R\downarrow\rangle\langle R\downarrow|-|L\downarrow\rangle\langle L\downarrow|=\frac{1}{2}(1+\sigma_3), \tag{33}$$

that is, the spin-down components are projected out regardless of the direction of propagation of the neutron. In this case, after frequent measurements E_1 performed at time intervals T/N, the evolution operator in Eq. (6) reads

$$V_N(T)=(E_1e^{-iHT/N}E_1)^N=E_1(t_\uparrow+r_\uparrow\tau_1)^N, \tag{34}$$

where $t_\uparrow\sim 1-igT/N$ and $r_\uparrow\sim -i\alpha gT/N$ for large N [see Eq. (26)]. Taking the limit, one obtains the following expression for the QZE evolution operator defined in Eq. (8):

$$\mathcal{V}(T)=\lim_{N\to\infty}V_N(T)=e^{-igT}E_1e^{-i\alpha gT\tau_1}. \tag{35}$$

Interesting physical situations can now be investigated. Choose, for instance, $gT=\pi$, $\alpha=-1/2$, $\beta=-1$, which belongs to case (i) in Eq. (30) [so that, without measurements, the neutron is totally transmitted with its spin flipped, as shown in Eq. (32)]. When the direction-insensitive measurements are continuously performed, the QZE evolution is $\mathcal{V}(T)=-iE_1\tau_1$ and the final state is

$$\mathcal{V}(T)|R\uparrow\rangle=-i|L\uparrow\rangle, \tag{36}$$

i.e., *the neutron spin is not flipped, but the neutron itself is totally reflected*. This clearly shows that reflection "losses" can be very important; as a matter of fact, reflection effects *dominate*, in this example. Notice that this is always an example of QZE: The projection operator E_1 in Eq. (33) *prevents* the spin from flipping. The point here is, however, that E_1 is not "tailored" so as to prevent the wave function from being reflected.

B. Another particular case: Seminal model

Let us now focus on a model corresponding to case (ii) in Eq. (31). The choice of parameters, e.g., $gT=\pi/2$, $\alpha=2n$, $\beta=-1$, obviously fulfills these conditions for arbitrary integer n. Total transmission with spin flipped occurs again when no measurement is performed.

When direction-insensitive spin measurements, described by projections E_1, are performed at time intervals T/N, the QZE evolution operator in Eq. (35) becomes, in the $N\to\infty$ limit, simply $\mathcal{V}_1(T)=-i(-1)^nE_1$ and the final state is

$$\mathcal{V}_1(T)|R\uparrow\rangle=-i(-1)^n|R\uparrow\rangle, \tag{37}$$

so that the "usual" QZE is obtained. When $n=0$ this is our seminal model [8], reviewed in Sec. III. Obviously, the case $n=0$ is not rich enough to yield information about reflection effects. In the following subsection the case of nonzero n will be discussed.

C. Direction-sensitive spin measurements

We now consider a different type of spin measurement. Let the measurement be characterized by the following projection operator:

$$E_2=1-|R\downarrow\rangle\langle R\downarrow|, \tag{38}$$

which projects out those neutrons that are transmitted with their spin flipped. Notice that spin-down neutrons that are reflected back are not projected out by E_2: for this reason we call this a "direction-sensitive" spin measurement. Refer to Fig. 3(b). Even though the action of this projection is not easy to implement experimentally, this example will clearly illustrate some interesting issues related to the Misra-Sudarshan theorem. Incidentally, we would like to point out that although the projection operator (38) looks rather artificial, its experimental realization is not impossible, at least in principle. Consider again the ISIS experiment [15,16], shortly mentioned after Eq. (22): In such a case, since the position of the neutron burst in the storage crystal is known with excellent approximation at any given time, one simply has to "switch on" the polarizer when the neutron burst impinges from the left, and to "switch it off" when the burst comes from right. It goes without saying that the model is consistent (and our discussion meaningful) because the free Hamiltonian does not play any role in our description and can be dropped from the outset.

We shall see that the action of the projector E_2 will yield a very interesting result. For large N, the evolution is given by

$$V_{2,N}(T)=(E_2e^{-iHT/N}E_2)^N=e^{-igT}\left(1-i\frac{gT}{N}Z\right)^NE_2+O(1/N), \tag{39}$$

where $Z\equiv E_2(H/g-1)E_2$.

The QZE evolution is given by the limit

$$\mathcal{V}_2(T)=\lim_{N\to\infty}V_{2,N}(T)=e^{-igT}e^{-igTZ}E_2. \tag{40}$$

To compute its effect on the initial state $|R\uparrow\rangle$, we note that, when acting on states $|R\uparrow\rangle$, $|L\uparrow\rangle$, and $|L\downarrow\rangle$, which span the "survival" subspace, the Z operator behaves as

$$Z\begin{pmatrix}|R\uparrow\rangle\\|L\uparrow\rangle\\|L\downarrow\rangle\end{pmatrix}=\begin{pmatrix}0&\alpha&\alpha\beta\\\alpha&0&\beta\\\alpha\beta&\beta&0\end{pmatrix}\begin{pmatrix}|R\uparrow\rangle\\|L\uparrow\rangle\\|L\downarrow\rangle\end{pmatrix}. \tag{41}$$

Let us choose for definiteness $\beta=-1$, so that

$$(Z-1/2)^2|R\uparrow\rangle=\theta^2|R\uparrow\rangle, \tag{42}$$

with $\theta=\sqrt{8\alpha^2+1}/2$. Thus the final state can be readily obtained,

$$\mathcal{V}_2(T)|R\uparrow\rangle=e^{-3igT/2}\left[\left(\cos(gT\theta)+\frac{i}{2\theta}\sin(gT\theta)\right)|R\uparrow\rangle+\frac{i\alpha}{\theta}\sin(gT\theta)(|L\downarrow\rangle-|L\uparrow\rangle)\right]. \tag{43}$$

Therefore, for a continuous direction-sensitive (namely, E_2) measurement, the probability of finding the initial state $|R\uparrow\rangle$ is not unity. Part of the wave function will be reflected, although the neutron would have been totally transmitted without measurement [see Eq. (32)] or with an "E_1 measurement" [see Eq. (37)].

Clearly, the action of the projector E_2 yields a completely different result from that of E_1 in Eq. (37). This is obvious and easy to understand: the state (43) belongs to the subspace of the "survived" states, *according to the projection* E_2. Notice also that the probability loss due to the measurements is zero, in the limit, because the QZE evolution (40) is unitary within the subspace of the "survived" states.

VI. A MORE REALISTIC MODEL

Let us now introduce a more realistic (albeit static) model. Such a model can be shown to be derivable from a Hamil-

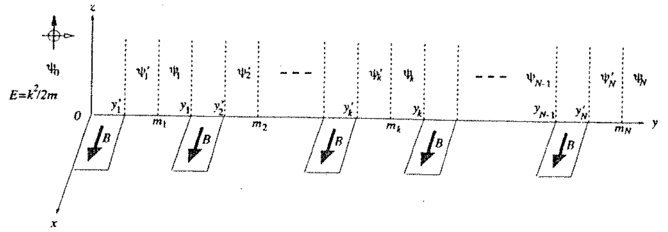

FIG. 4. Spin-up neutron moving along the $+y$ direction with energy E. The magnetic field points to the $+x$ direction and is zero in the region between y_n' and y_n, in which the measurements will be made. In these field-free regions the wave functions are $|\psi_n'\rangle$ before measurement and $|\psi_n\rangle$ after the measurement.

tonian very similar to the one studied in the previous sections by a suitable identification of parameters (see the Appendix). The effect of reflections in the QZE will now be tackled by directly solving a stationary Schrödinger equation, which will be set up as follows.

Let a neutron with energy $E=k^2/2m$ and spin up ($+z$ direction), moving along the $+y$ direction, impinge on N regions of constant magnetic field pointing to the x direction, among which there are $N-1$ field-free regions. The thickness of a single piece of magnetic field is a and the field-free region has size b. The configuration is shown in Fig. 4. Thus we have the one-dimensional scattering problem of a neutron off a piecewise constant magnetic field with total thickness $D=Na$. The stationary Schrödinger equation is described by the Hamiltonian

$$H_Z=\frac{p_y^2}{2m}+\mu B\sigma_1\Omega(y), \tag{44}$$

where μ is the modulus of the neutron magnetic moment, B the strength of the magnetic field, and

$$\Omega(y)=\begin{cases}0 & \text{for} \quad y<0, \quad y_n'<y<y_n, \quad y_N'<y \quad (n=1,2,\ldots,N),\\ 1 & \text{for} \quad y_{n-1}<y<y_n' \quad (n=1,2,\ldots,N),\end{cases} \tag{45}$$

with $y_n=n(a+b)$ and $y_n'=y_{n-1}+a$, characterizes the configuration of the magnetic field B applied along the x axis. Refer to Fig. 4. The incident state of the neutron is taken to be $|\psi_{\rm in}\rangle=e^{iky}|\uparrow\rangle$. Let $r_{\uparrow(\downarrow)}$ be the reflection amplitude for the spin-up (spin-down) component. The wave function for $y<0$ is written as

$$|\psi_0\rangle=e^{iky}|\uparrow\rangle+e^{-iky}[r_\uparrow|\uparrow\rangle+r_\downarrow|\downarrow\rangle]. \tag{46}$$

Denoting the transmission amplitudes for spin-up and spin-down as $t_\uparrow$ and $t_\downarrow$, the outgoing wave function in the region $y>y_N'$ reads

$$|\psi_N\rangle=e^{iky}[t_\uparrow|\uparrow\rangle+t_\downarrow|\downarrow\rangle]. \tag{47}$$

Since $[\sigma_1,H_Z]=0$, it is convenient to work with the basis $|\pm\rangle=(|\uparrow\rangle\pm|\downarrow\rangle)/\sqrt{2}$, i.e., the eigenstates of σ_1 belonging to eigenvalues ± 1. For later use we denote $r_\pm=r_\uparrow\pm r_\downarrow$ and $t_\pm=t_\uparrow\pm t_\downarrow$.

In the field-free region, before the point $y=m_n$ where the nth measurement is assumed to take place, $y_n'<y<m_n$, the wave function is

$$|\psi_n'\rangle=\sum_{\sigma=\pm}(R_{n,\sigma}'e^{ik(y-y_n')}+L_{n,\sigma}'e^{-ik(y-y_n')})|\sigma\rangle \qquad (n=1,2,\ldots,N). \tag{48}$$

On the other hand, in the region after the nth measurement, $m_n<y<y_n$, the wave function is

$$|\psi_n\rangle=\sum_{\sigma=\pm}(R_{n,\sigma}e^{ik(y-y_n)}+L_{n,\sigma}e^{-ik(y-y_n)})|\sigma\rangle \qquad (n=0,1,\ldots,N). \tag{49}$$

The relation between the amplitudes of the wave functions $|\psi_{n+1}'\rangle$ and $|\psi_n\rangle$ at the right- and left-hand sides of the nth potential region is determined by the boundary conditions at points y_n and y_n'. In fact, we have

$$\begin{pmatrix}R_{n+1,\pm}'\\ L_{n+1,\pm}'\end{pmatrix}=M_\pm\begin{pmatrix}R_{n,\pm}\\ L_{n,\pm}\end{pmatrix}, \tag{50}$$

where the transfer matrix is given by

$$M_\pm=\begin{pmatrix}\cos k_\pm a+i\cosh\eta_\pm\sin k_\pm a & -i\sinh\eta_\pm\sin k_\pm a\\ i\sinh\eta_\pm\sin k_\pm a & \cos k_\pm a-i\cosh\eta_\pm\sin k_\pm a\end{pmatrix} \tag{51}$$

with $k_\pm=\sqrt{k^2\mp 2m\mu B}$ and $k/k_\pm=e^{\eta_\pm}$. This can also be expressed in a concise way in terms of the Pauli matrices in this two-dimensional space, τ_1, τ_2, and τ_3, as

$$M_\pm=e^{\eta_\pm(1+\tau_1)/2}e^{ik_\pm a\tau_3}e^{-\eta_\pm(1+\tau_1)/2}. \tag{52}$$

We clearly see that the above formula contains all the boundary information at points y_n and y_n': The first and last factors are the kicks exerted at the boundaries of a single piece of constant magnetic field, while the central one represents a free evolution with relative energy $E\mp\mu B$.

In what follows, we shall incorporate the measurement processes performed at points m_n as some kind of boundary conditions, connecting the primed and unprimed wave functions in the field-free region.

A. Evolution without any spin measurements

We first consider the case where there is no measurement at all. This enables us to set up the notation and rederive some known results (which will be useful for future comparison). In this case the primed and unprimed wave functions must be equal $|\psi_n'\rangle=|\psi_n\rangle$ in the field-free region. By virtue of Eq. (50) we obtain

$$\begin{pmatrix} R_{n+1,\pm} \\ L_{n+1,\pm}\end{pmatrix}=e^{ikb\tau_3}M_\pm\begin{pmatrix} R_{n,\pm} \\ L_{n,\pm}\end{pmatrix}. \tag{53}$$

Notice the boundary conditions $R_{0,\pm}=1/\sqrt{2}$ and $L_{N,\pm}=0$ together with the definitions of transmission amplitude $R_{N,\pm}=e^{iky_N}t_\pm/\sqrt{2}$ and reflection amplitude $L_{0,\pm}=r_\pm/\sqrt{2}$. After applying the above equation N times, we obtain the following relation:

$$e^{iky_N}\begin{pmatrix} t_\pm \\ 0\end{pmatrix}=([N]_\pm e^{ikb\tau_3}M_\pm-[N-1]_\pm)\begin{pmatrix} 1 \\ r_\pm\end{pmatrix}, \tag{54}$$

where $[N]_\pm=(q_\pm^N-q_\pm^{-N})/(q_\pm-q_\pm^{-1})$, with $q_\pm, q_\pm^{-1}$ being the two eigenvalues of the transfer matrix $e^{ikb\tau_3}M_\pm$, which are determined by

$$\frac{q_\pm+q_\pm^{-1}}{2}=\cos kb\cos k_\pm a-\cosh\eta_\pm\sin kb\sin k_\pm a. \tag{55}$$

When there is only a single piece of magnetic field with length a, i.e., $N=1$, the transmission amplitude of a neutron in the spin state $|\pm\rangle$ is

$$t_{a\pm}=\frac{e^{-ika}}{\cos k_\pm a-i\cosh\eta_\pm\sin k_\pm a}, \tag{56}$$

as is well known. From Eq. (54), for an arbitrary $N>1$, the transmission amplitude of the same neutron passing through a magnetic field with a latticelike structure as depicted in Fig. 4 can be written as

$$t_\pm=\frac{e^{-iky_N}t_{a\pm}}{e^{-iky_1}[N]_\pm-[N-1]_\pm t_{a\pm}}. \tag{57}$$

For a neutron in its spin-up state $|\uparrow\rangle$, the transmission amplitude with spin unflipped is then $t_\uparrow=(t_++t_-)/2$ and that with spin flipped is $t_\downarrow=(t_+-t_-)/2$. As a result, for a spin-up neutron to go through a constant potential of width $y_N=D=Na$ without reflection and with spin flipped, i.e., $|t_\downarrow|=1$, one should require $k_\pm D=n_\pm\pi$ or

$$E=\frac{\pi^2(n_+^2+n_-^2)}{4mD^2},\quad \mu B=\frac{\pi^2(n_+^2-n_-^2)}{4mD^2}, \tag{58}$$

with $n_\pm$ two arbitrary integers, their difference n_+-n_- being an odd number. In this case of complete transmission $|t_\downarrow|=1$, the energy E must be larger than the potential μB. The rest of the analysis above, however, is valid also when the energy is less than the potential.

Now we consider the case where N tends to infinity and the magnetic field possesses a periodic lattice structure. The relation (53) still holds and in order to preserve the translational symmetry along the y axis [that is, to keep the Hamiltonian invariant under a translation of $(a+b)$ along the y axis], one should have $|q_\pm|=1$ owing to the Bloch theorem. Equivalently, the trace of the transfer matrix $e^{ikb\tau_3}M_\pm$ as given in Eq. (55) should not be greater than 1. This determines the energy band of the system: those energies that make the absolute value of this trace greater than 1 will be forbidden, because for these energies $|q_\pm|$ or $|q_\pm|^{-1}$ becomes larger than 1 and $[N]_\pm$ tends exponentially to infinity when N approaches infinity. For large N, even if there is no periodical structure, there is always some k that makes this trace greater than 1 (e.g., $kb+k_\pm a=l\pi$). Therefore, the transmission probability will tend to zero exponentially when N becomes large, even though the energy may be very large relative to the potential. This shows that reflection effects in the presence of a lattice structure are very important; as we shall see, this feature is preserved even when projection operators are interspersed in the lattice.

B. Direction-insensitive projections

We consider now the second situation, when direction-insensitive measurements are performed at points m_n's. By this kind of measurement, the spin-down components are projected out and the spin-up components evolve freely regardless whether the neutron is traveling right or left.

The boundary conditions imposed by this kind of measurement at point m_n for the wave function $|\psi_n\rangle$ and $|\psi_n'\rangle$ in the field-free region are expressed as

$$R_{n,\downarrow}=L'_{n,\downarrow}=0,\quad \begin{pmatrix} R'_{n,\uparrow} \\ L'_{n,\uparrow}\end{pmatrix}=e^{-ikb\tau_3}\begin{pmatrix} R_{n,\uparrow} \\ L_{n,\uparrow}\end{pmatrix}, \tag{59}$$

where $R_{n,\uparrow}=(R_{n,+}+R_{n,-})/\sqrt{2}$ and $R_{n,\downarrow}=(R_{n,+}-R_{n,-})/\sqrt{2}$ for right-going components and similar expressions for the left-going and primed components. Therefore, application of Eq. (50) N times yields

$$\begin{pmatrix} R_{N,\uparrow} \\ L_{N,\uparrow}\end{pmatrix}=(e^{ikb\tau_3}M_1)^N\begin{pmatrix} R_{0,\uparrow} \\ L_{0,\uparrow}\end{pmatrix}, \tag{60}$$

where the 2×2 transfer matrix M_1 has the following matrix elements:

$$(M_1)_{ij}=\bar{M}_{ij}-\Delta M_{i2}\Delta M_{2j}/\bar{M}_{22} \tag{61}$$

with $\bar{M}=(M_+ + M_-)/2$ and $\Delta M=(M_+ - M_-)/2$.

Now we take the limit as required by a "continuous" measurement, i.e., $N\to\infty$, $a\to 0$ keeping $Na=D$ finite and $Nb\to 0$. By the definition (51) of the transfer matrix, we have the small-a expansions

$$\bar{M}=1+ika\tau_3+O(a^2), \quad \Delta M=\zeta ka(\tau_2-i\tau_3)+O(a^2) \tag{62}$$

with $\zeta\equiv\mu B/2E$, obtaining

$$\lim_{N\to\infty}(e^{ikb\tau_3}M_1)^N=e^{ikD\tau_3}. \tag{63}$$

Recall that $t_\uparrow=e^{-ikD}R_{N,\uparrow}$ is the transmission amplitude, $L_{0,\downarrow}=r_\downarrow$ the reflection amplitude, and $L_{N,\uparrow}=0$ and $R_{0,\uparrow}=1$ because of the boundary conditions. After taking the limit $N\to\infty$ in Eq. (60), we see that the transmission (survival) probability becomes 1, i.e., $|t_\uparrow|^2=1$, for *any* input energy and magnetic field. This reveals another aspect of neutron QZE: When the energy of the neutron is smaller than the potential, the transmission probability decays exponentially when the length increases and no measurement is performed; by contrast, when continuous direction-insensitive measurements are made, one can obtain a total transmission.

If we choose the energy of the neutron and the potential as in Eq. (58), without measurements the neutron will be totally transmitted with its spin flipped. On the other hand, if the spin-up state is measured continuously, the neutron will be totally transmitted with its spin unflipped. This is exactly the QZE in the usual sense. Our analysis enables us to see that two kinds of QZEs are taking place: One is the QZE for the right-going neutron, by which we obtain a total transmission of the right-going input state, and another one is for the left-going neutron, which preserves the zero amplitude of the left-going input state. This case corresponds to projector E_1 in our simplified model in Sec. V B.

C. Direction-sensitive projections

The third case we consider is the direction-sensitive measurement. By this kind of measurement the left-going components (or the reflection parts) evolve freely, no matter whether spin is up or down, and the right-going components are projected to the spin-up state. The corresponding boundary conditions are

$$R_{n,\downarrow}=0, \quad L_{n,\pm}=e^{-ikb}L'_{n,\pm}. \tag{64}$$

If we apply Eq. (50) N times, supplemented with these boundary conditions, the following relations among the transmission and reflection amplitudes are obtained:

$$e^{ikD}\begin{pmatrix} t_\uparrow \\ 0 \\ 0 \end{pmatrix}=(e^{ikb\Sigma_3}M_2)^N\begin{pmatrix} 1 \\ r_\uparrow \\ r_\downarrow \end{pmatrix}, \tag{65}$$

where Σ_3 is a diagonal matrix $\Sigma_3=\mathrm{diag}\{1,-1,-1\}$ and the 3×3 transfer matrix M_2 is given by

$$M_2=\begin{pmatrix} \bar{M}_{11} & \bar{M}_{12} & \Delta M_{12} \\ \bar{M}_{21} & \bar{M}_{22} & \Delta M_{22} \\ \Delta M_{21} & \Delta M_{22} & \bar{M}_{22} \end{pmatrix}. \tag{66}$$

In the limit of continuous measurements ($N\to\infty$, $a\to 0$, while keeping $D=Na$ constant, and $Nb\to 0$), the transfer matrix is expanded as

$$M_2=1-ika/3+ikaZ_2+O(a^2), \tag{67}$$

for small a, with ($\zeta=\mu B/2E$)

$$Z_2\equiv\begin{pmatrix} 4/3 & 0 & -\zeta \\ 0 & -2/3 & \zeta \\ \zeta & \zeta & -2/3 \end{pmatrix}, \tag{68}$$

and we have

$$\lim_{N\to\infty}(e^{ikb\Sigma_3}M_2)^N=e^{-ikD/3}e^{ikDZ_2}. \tag{69}$$

Notice that the matrix Z_2 satisfies $\Sigma_3 Z_2\Sigma_3=Z_2^\dagger$, from which we obtain, in the above limit, the conservation of probability

$$|t_\uparrow|^2+|r_\uparrow|^2+|r_\downarrow|^2=1. \tag{70}$$

This shows that there are no losses caused by the continuous direction-sensitive measurements. On the other hand, the transmission amplitude with spin unflipped is explicitly given by

$$t_\uparrow=\frac{e^{-i4kD/3}}{(e^{-ikDZ_2})_{11}}, \tag{71}$$

which implies that the transmission probability $|t_\uparrow|^2$ is in general *not* equal to 1. To have a general impression of its behavior, we plot $T_\uparrow=|t_\uparrow|^2$ as a function of kD and ζ in Fig. 5.

Some comments are in order. There are two critical values for ζ, namely 0 and $\zeta_c=4\sqrt{3}/9\approx 0.77$. When $0\le\zeta<\zeta_c$, the matrix Z_2 has three real eigenvalues and the transmission probability will oscillate depending on kD. When $\zeta=\zeta_c$ the transmission probability will decay according to $(kD)^{-2}$. In fact, if one defines $G=Z_2-2/3$, it is easy to show that $e^{-ikDG}=1-ikDG+(e^{2ikD}-1-2ikD)G^2/4$, because G satisfies $G^2(G+2)=0$. Then one can explicitly confirm that the element $(e^{-ikDG})_{11}$ includes a linear kD term, which gives the $(kD)^{-2}$ behavior to the transmission probability. Finally, when $\zeta>\zeta_c$ the matrix Z_2 has two imaginary eigenvalues and therefore the transmission probability decays exponentially with kD. This can be seen clearly in Fig. 5(a). An interesting case arises when we consider $1/2<\zeta<\zeta_c$ or $E<\mu B<8\sqrt{3}E/9\approx 1.5E$. Without measurements, the transmission probability decays exponentially when the length of the magnetic field is increased, because the input energy is smaller than the potential. When continuous measurements are performed, however, the transmission probability will oscillate as the length of the magnetic field increases.

As we can see in Fig. 6, although the conditions (58) for total transmission in the absence of measurements have been

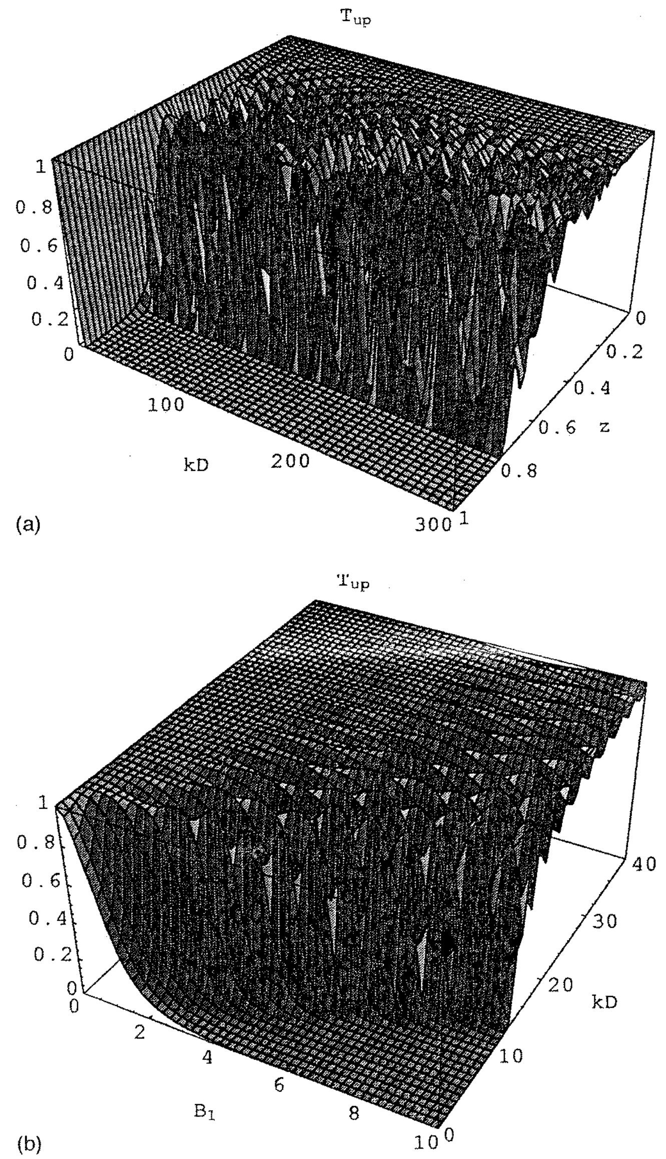

FIG. 5. Transmission probability with spin unflipped $T_{\text{up}} = |t_\uparrow|^2$ is plotted as a function of kD and $z=\zeta$ in (a) and as a function of $B_1 = \sqrt{m\mu B}D$ and kD in (b).

imposed, the transmission probability $T_\uparrow$ is not 1, as it would be for the "ordinary" QZE. Reflections are unavoidable. This case corresponds to the projector E_2 considered in the simplified model.

As we have seen, there are peculiar reflection effects in the presence of projections, when D (total length) is varied. This is clearly an interference effect, which can lead to enhancement of reflection "losses," if the "projection" does not suppress the left component of the wave (this is what happens for E_2). This proves that reflection effects can become very important in experimental tests of the QZE with neutron spin, if, roughly speaking, the total length of the interaction region "resonates" with the neutron wavelength. It is interesting that such a resonance effect takes place even though the dynamical properties of the system are profoundly modified by the projection operators, in the limit of "continuous" measurements, leading to the QZE.

Finally, we would like to stress again that we are performing an analysis in terms of stationary states (i.e., transmission/reflection coefficients for plane waves), while at the same time we are analyzing a quantum Zeno phenomenon, which is essentially a time-dependent effect. This is meaningful within our approximations, where the wave-packet spread is neglected and the measurements are performed with very high frequency. A more sophisticated argument in support of this view is given in the Appendix. In the present context, wave-packet effects, if taken into account, would result in a sort of average of the effects shown in Figs. 5 and 6 (which refer to the monochromatic case); however, our general conclusions would be unaltered. It is worth stressing that, in neutron optics, effects due to a high sensitivity to fluctuation phenomena (such as fluctuations of the intensity of the magnetic field) become important at high wave number and constitute an experimental challenge [18].

VII. SUMMARY

We have analyzed some peculiar features of a quantum Zeno-type dynamics by discussing the noteworthy example of a neutron spin evolving under the action of a magnetic filed in the presence of different types of measurements ("projections").

The "survival probability" depends on our definition of "surviving," i.e., on the choice of the projection operator E. Different E's will yield different final states, and Misra and Sudarshan's theorem [2] simply makes sure that the survival probability is unity: the final state belongs to the subspace of the survived products.

In the physical case considered (neutron spin), our examples clarify that the practical details of the experimental procedure by which the neutron spin is "measured" are very important. For example, in order to avoid constructive interference effects, leading to (unwanted) enhancement of the reflected neutron wave, it is important to devise the experimental setup in such a way that reflection effects are suppressed.

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APPENDIX

In this appendix, we shall endeavor to establish a connection between the models analyzed in Secs. IV and VI. In other words, we will examine whether the parametrization of the Hamiltonian of the form (22) is compatible with the more realistic one considered in Eq. (44) and in such a case find which values are to be assigned to the parameters α, β, and g. To this end, it is enough to consider the scattering (i.e., the transmission and reflection) process of a neutron off a single constant magnetic field B of width a. We compare the scattering amplitudes calculated on the basis of the simple abstract Hamiltonian (22) and of the more realistic one (44).

Notice that the process is treated as a dynamical one in the former case (T is regarded, roughly speaking, as the time necessary for the neutron to go through the potential), while in the latter case we treat it as a stationary problem.

Observe first that the tranfer matrix $M_\pm$ in Eq. (51), derived for the stationary scattering process, yields the following transmission/reflection amplitudes:

$$\begin{aligned} R'_{1,\uparrow}&=\frac{1}{2}\left(\frac{1}{(M_+)_{22}}+\frac{1}{(M_-)_{22}}\right),\\ R'_{1,\downarrow}&=\frac{1}{2}\left(\frac{1}{(M_+)_{22}}-\frac{1}{(M_-)_{22}}\right),\\ L_{0,\uparrow}&=-\frac{1}{2}\left(\frac{(M_+)_{21}}{(M_+)_{22}}+\frac{(M_-)_{21}}{(M_-)_{22}}\right),\\ L_{0,\downarrow}&=-\frac{1}{2}\left(\frac{(M_+)_{21}}{(M_+)_{22}}-\frac{(M_-)_{21}}{(M_-)_{22}}\right). \end{aligned} \tag{A1}$$

It is easy to show that the relations (A1) are equivalent to

$$\begin{pmatrix}1&1&1&1\\1&-1&1&-1\\1&1&-1&-1\\1&-1&-1&1\end{pmatrix}\begin{pmatrix}R'_{1,\uparrow}\\R'_{1,\downarrow}\\L_{0,\uparrow}\\L_{0,\downarrow}\end{pmatrix}=\begin{pmatrix}\mathcal{M}_{-,+}\\\mathcal{M}_{-,-}\\\mathcal{M}_{+,+}\\\mathcal{M}_{+,-}\end{pmatrix}, \tag{A2}$$

where we have introduced

$$\mathcal{M}_{+,\pm}=\frac{1+(M_\pm)_{21}}{(M_\pm)_{22}},\quad \mathcal{M}_{-,\pm}=\frac{1-(M_\pm)_{21}}{(M_\pm)_{22}}. \tag{A3}$$

It is important to realize that these quantities are just phase factors. In fact, since

$$(M_\pm)_{21}=i\sinh\eta_\pm\sin k_\pm a$$

and

$$(M_\pm)_{22}=\cos k_\pm a-i\cosh\eta_\pm\sin k_\pm a \tag{A4}$$

and

$$|1\pm(M_\pm)_{21}|^2=|(M_\pm)_{22}|^2=1+\sinh^2\eta_\pm\sin^2 k_\pm a, \tag{A5}$$

their absolute values are unity. Thus we can rewrite them in the form

$$\mathcal{M}_{+,\pm}=e^{i(\xi_\pm+\phi_\pm)},\quad \mathcal{M}_{-,\pm}=e^{i(-\xi_\pm+\phi_\pm)}, \tag{A6}$$

where

$$\xi_\pm=\tan^{-1}(\sinh\eta_\pm\sin k_\pm a)$$

and

$$\phi_\pm=\tan^{-1}(\cosh\eta_\pm\tan k_\pm a). \tag{A7}$$

Observe now that Eq. (27), dynamically derived from the abstract Hamiltonian (22), is equivalent to

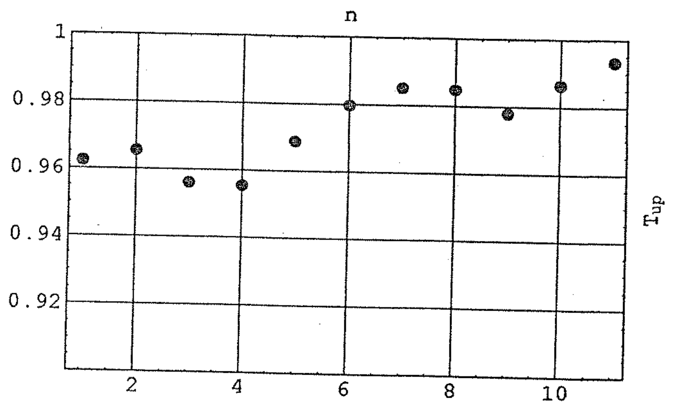

FIG. 6. The transmission probablity with spin unflipped $T_\uparrow=|t_\uparrow|^2$ as a function of n, when the conditions (58) for total transmission are satisfied with $n_-=n$ and $n_+=n+9$.

$$\begin{pmatrix}t_\uparrow\\t_\downarrow\\r_\uparrow\\r_\downarrow\end{pmatrix}=\frac{1}{4}\begin{pmatrix}1&1&1&1\\1&-1&1&-1\\1&1&-1&-1\\1&-1&-1&1\end{pmatrix}\begin{pmatrix}e^{-iE_{++}T}\\e^{-iE_{+-}T}\\e^{-iE_{-+}T}\\e^{-iE_{--}T}\end{pmatrix}. \tag{A8}$$

The apparent similarity between the above relation and Eq. (A2), valid in the stationary scattering setup, induces us to look for a more definite connection between the two cases.

If we slightly generalize the abstract Hamiltonian (22),

$$H_{\rm dyn}=g[1+\alpha\tau_1+\beta\sigma_1+\gamma\tau_1\sigma_1], \tag{A9}$$

by introducing the additional parameter γ, we easily find the correspondence existing between the parameters involved: The incident wave number k of the neutron and the configuration of the static potential (strength B and width a) determine the scattering data, which are reproducible by an appropriate choice of parameters α, β, γ, and gT in the dynamical process governed by the Hamiltonian (A9).

For definiteness, consider the case of narrow potential, that is, $a\to 0$ or $ka\ll 1$. Incidentally, notice that this is the case of interest for the analysis of the QZE. The above $\xi_\pm$ and $\phi_\pm$ are then approximated as

$$\xi_\pm\sim\pm\zeta ka,\quad \phi_\pm\sim(1\mp\zeta)ka, \tag{A10}$$

where we set $\zeta=\mu B/2E=m\mu B/k^2$, as in Sec. VI. In the limit $a\to 0$, the evolution time T is also considered to be of the same order of a and the transmission and reflection coefficients are expressed, in terms of the parameters α, β, γ, and gT, as

$$\begin{pmatrix}t_\uparrow\\t_\downarrow\\r_\uparrow\\r_\downarrow\end{pmatrix}\sim\begin{pmatrix}1\\-i\beta gT\\-i\alpha gT\\-i\gamma gT\end{pmatrix}. \tag{A11}$$

In the stationary scattering problem, the same quantities are calculated to be

$$\begin{pmatrix} t_\uparrow \\ t_\downarrow \\ r_\uparrow \\ r_\downarrow \end{pmatrix} = \begin{pmatrix} e^{-ika}R'_{1,\uparrow} \\ e^{-ika}R'_{1,\downarrow} \\ L_{0,\uparrow} \\ L_{0,\downarrow} \end{pmatrix} \sim \begin{pmatrix} 1-ika+i(\phi_+ + \phi_-)/2 \\ i(\phi_+ - \phi_-)/2 \\ -i(\xi_+ + \xi_-)/2 \\ -i(\xi_+ - \xi_-)/2 \end{pmatrix} \sim \begin{pmatrix} 1 \\ -i\zeta ka \\ 0 \\ -i\zeta ka \end{pmatrix}. \tag{A12}$$

Therefore, the abstract Hamiltonian

$$H_{\rm dyn} = \mu B(1+\tau_1)\sigma_1 \tag{A13}$$

can reproduce the desired scattering data when the system evolves under this Hamiltonian for time $T = a/v = ma/k$.

It is also interesting to see how such a dynamical Hamiltonian $H_{\rm dyn}$ may reproduce the transfer matrix $M_\pm$ (51), which further confirms the equivalence between the two formalisms, stationary and dynamical, governed by the Hamiltonians H_Z and $H_{\rm dyn}$, respectively. For this purpose, consider first a neutron, initially prepared in state $|R\pm\rangle$, subject to the dynamical evolution engendered by $H_{\rm dyn}$ for time $T = ma/k$. By definition, the transfer matrix connects the scattering products in the following way:

$$\begin{pmatrix} R'_{1,\pm} \\ 0 \end{pmatrix} = M_\pm \begin{pmatrix} 1 \\ L_{0,\pm} \end{pmatrix}. \tag{A14}$$

These scattering amplitudes are given by the corresponding matrix elements of the evolution operator e^{-iHT},

$$e^{-ika}R'_{1,\pm} = \langle R\pm|e^{-iHT}|R\pm\rangle, \quad L_{0,\pm} = \langle L\pm|e^{-iHT}|R\pm\rangle, \tag{A15}$$

which reduces, for small T, to

$$R'_{1,\pm} \sim 1 + ika \mp i\mu BT, \quad L_{0,\pm} \sim \mp i\mu BT. \tag{A16}$$

On the other hand, if a neutron is prepared in $|L\pm\rangle$, we have the relation

$$\begin{pmatrix} R'_{1,\pm} \\ e^{-ika} \end{pmatrix} = M_\pm \begin{pmatrix} 0 \\ L_{0,\pm} \end{pmatrix}, \tag{A17}$$

where

$$\begin{aligned} R'_{1,\pm} &= e^{ika}\langle R\pm|e^{-iHT}|L\pm\rangle \sim \mp i\mu BT, \\ L_{0,\pm} &= \langle L\pm|e^{-iHT}|L\pm\rangle \sim 1 \mp i\mu BT. \end{aligned} \tag{A18}$$

It is now an easy task to determine the matrix elements of $M_\pm$ from the above relations (A14)–(A18). We obtain

$$\begin{aligned} M_\pm &\sim \begin{pmatrix} 1+ika\mp i\mu BT & \mp i\mu BT \\ \pm i\mu BT & 1-ika\pm i\mu BT \end{pmatrix} \\ &= 1 - i[\pm\mu B(i\tau_2+\tau_3) - 2E\tau_3]T. \end{aligned} \tag{A19}$$

By defining a "generator" G_d,

$$G_d = \mu B(i\tau_2+\tau_3)\sigma_1 - 2E\tau_3, \tag{A20}$$

the transfer matrix $M_\pm$ for finite a (or T) can be rewritten as

$$M_\pm = \langle\pm|e^{-iG_dT}|\pm\rangle, \tag{A21}$$

which is nothing but the transfer matrix (51), obtained for the stationary-state problem from the Hamiltonian H_Z.

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Two-Level System with a Noisy Hamiltonian

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We study the dynamical properties of a two-level system in interaction with its environment, whose action on the system is modeled by means of a noise term in the Hamiltonian. We solve the Schrödinger equation, obtain an evolution equation of the Lindblad type for the noise average of the density matrix, and discuss the results in terms of a "decoherence parameter." Finally, we concentrate our attention on the possibility of hindering the transitions between the two levels in two (apparently unrelated) ways: (a) by increasing the strength of the noise; (b) by a series of frequent measurements. There is an interesting relation between these two situations.

KEY WORDS: Decoherence; stochastic; two-level system.

To model the interaction between a quantum mechanical system and its environment is in general a complicated problem. A possible approach is to represent the action of the environment via "noise" terms in the Hamiltonian of the system; however, it is not clear whether there is a general recipe in order to get such terms from the total Hamiltonian (describing the environment + the system) in a rigorous way. As a matter of fact, this program can be carried out only for some solvable models. Interesting examples are the Ford–Kac–Mazur [1] and a whole class of related models [2] that have played an important role in clarifying several aspects related to dissipative phenomena. In these cases, an ensemble of coupled oscillators interacts via a quadratic Hamiltonian and the reduced dynamics of one of these oscillators yields, in an appropriate macroscopic limit, a Langevin equation [3]. The noise terms are therefore "derived" from the total Hamiltonian in some approximation and under some assumptions, by formally solving the equations of motion. Closely related examples are the Caldeira–Leggett [4] and the Feynman–Vernon model [5].

A somewhat more pragmatic approach consists of investigating the effect of noise without endeavoring to clarify its origin. In this case, the action of the environment on the system is schematized from the outset by means of a sort of stochastic operator added to the original Hamiltonian of the system.

The purpose of the present article is to scrutinize a simple model proposed by Blanchard, Bolz, Cini, De Angelis, and Serva (BBCDS) [6] and provide an explicit solution to discuss the short-time dynamics of a quantum system under the influence of a "noisy" environment, in relation to the so-called quantum Zeno phenomena. This Hamiltonian models a two-level system interacting with an environment, whose action on the system is simply schematized by means of a white noise multiplying an operator of the system. Its interest lies in the fact that the model can schematize a superconducting ring enclosing a quantized magnetic flux. Coherent tunneling between the two flux configurations is possible if the system is very well isolated from its environment. Due to the high sensitivity of the properties of such a device to its interaction with the environment, it becomes very important to model and analyze the action of the latter. The same model was also investigated by Berry [7], who concentrated his attention on the relative time scales as well as on the so-called quantum Zeno effect [8].

The plan of the article is as follows. In Section 1 we set up a general framework for our analysis.

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In Section 2 we introduce and analyze the BBCDS model, rederiving some results by an alternative method. We discuss our results in Section 3, by means of a "decoherence parameter." The quantum Zeno effect is discussed in Sections 4 and 5. It is shown that the decoherence caused by the interaction with the environment leads naturally to the exponential decay of the system and such behavior is in general at variance with the occurrence of the Zeno effect, even though the latter can be observed for a particular choice of the initial state. Section 6 is devoted to summary and comments.

1. THE GENERAL FRAMEWORK

We shall describe a system embedded in its environment via the time-dependent Hamiltonian

$$H = H_0 + \eta(t) H_1 \tag{1}$$

where H_0 and H_1 are Hermitian, time-independent operators. The action of the environment on the system is modeled by the stochastic term ηH_1, where η is a white noise, whose stochastic properties read

$$\langle \eta(t) \rangle = 0 \qquad \langle \eta(t)\eta(t') \rangle = \delta(t - t') \tag{2}$$

the brakets denoting an ensemble average over all possible realizations of the noise. The above formulas are recast into a more rigorous form by introducing the Wiener process:

$$\begin{aligned} dW(t) &\equiv W(t + dt) - W(t) = \int_t^{t+dt} \eta(s)\, ds \\ \langle dW(t) \rangle &= 0 \qquad \langle dW(t)\, dW(t) \rangle = dt \end{aligned} \tag{3}$$

The corresponding Schrödinger equation reads, by Ito calculus, ($\hbar = 1$)

$$\begin{aligned} |d\psi\rangle &= -iH_0|\psi\rangle\, dt - iH_1|\psi\rangle \circ dW \\ &= (-iH_0 - \tfrac{1}{2}H_1^2)|\psi\rangle\, dt - iH_1|\psi\rangle\, dW \end{aligned} \tag{4}$$

where $\circ$ denotes the Stratonovich product. Notice that, when the average (2)–(3) (to be denoted with a bar) is computed, the above equation reads

$$\overline{|d\psi\rangle} = (-iH_0 - \tfrac{1}{2}H_1^2)\, \overline{|\psi\rangle}\, dt \tag{5}$$

and probabilities are conserved, since

$$\overline{\| |\psi + d\psi\rangle \|^2} = \overline{\| |\psi\rangle \|^2} = 1 \tag{6}$$

The evolution of the density matrix is governed by

$$\begin{aligned} d\rho &= \rho + d\rho - \rho \\ &= |\psi + d\psi\rangle\langle\psi + d\psi| - |\psi\rangle\langle\psi| \\ &= -i[H_0, \rho]\, dt - \tfrac{1}{2}\{H_1^2, \rho\}\, dt \\ &\quad - i[H_1, \rho]\, dW + H_1\rho H_1\, dt \end{aligned} \tag{7}$$

where $[\cdot, \cdot]$ is the commutator and $\{\cdot, \cdot\}$ is the anticommutator. This yields, on the average, a Lindblad equation [9],

$$\frac{d}{dt}\bar{\rho} = -i[H_0, \bar{\rho}] - \frac{1}{2}\{H_1^2, \bar{\rho}\} + H_1\bar{\rho}H_1 \tag{8}$$

The operator H_1 is therefore the generator of a Gaussian semigroup, which in turn justifies Eq. (6). The Hamiltonian considered in [6,7] is a particular case of the above.

2. THE BBCDS MODEL AND THE ASSOCIATED LINDBLAD EQUATION

The BBCDS Hamiltonian describes a two-level system interacting with an environment,

$$H = \alpha\sigma_1 + \beta\eta(t)\sigma_3 \tag{9}$$

where α and β are real positive constants and σ_i ($i = 1,2,3$) Pauli matrices. The action of the environment on the system is modeled by the white noise η, and the corresponding Schrödinger equation reads

$$\begin{aligned} |d\psi\rangle &= -i\alpha\sigma_1|\psi\rangle\, dt - i\beta\sigma_3|\psi\rangle \circ dW \\ &= \left(-i\alpha\sigma_1 - \frac{1}{2}\beta^2\right)|\psi\rangle\, dt - i\beta\sigma_3|\psi\rangle\, dW \end{aligned} \tag{10}$$

where $|\psi\rangle = (|\psi_+\rangle, |\psi_-\rangle)^t$ is a two-component spinor. We shall work in the basis of the eigenstates of σ_3. Notice that when $\beta = 0$, the above equation yields coherent (Rabi) oscillations between the two eigenstates of σ_3.

By introducing the polarization (Bloch) vector

$$\mathbf{x}(t) = \langle\psi|\boldsymbol{\sigma}|\psi\rangle \tag{11}$$

one easily derives the stochastic differential equation

$$d\mathbf{x}(t) = A\mathbf{x}(t)\, dt + B\mathbf{x}(t)\, dW(t) \tag{12}$$

where

$$A = \begin{pmatrix} -2\beta^2 & 0 & 0 \\ 0 & -2\beta^2 & -2\alpha \\ 0 & 2\alpha & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & -2\beta & 0 \\ 2\beta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{13}$$

The third component $z = \langle\psi_+|\psi_+\rangle - \langle\psi_-|\psi_-\rangle$ of the Bloch vector contains information about the probability of finding the system in one of the eigenstates of σ_3. Observe that the norm of the Bloch vector is preserved,

$$\|\mathbf{x}(t)\|^2 \equiv x^2(t) + y^2(t) + z^2(t) = 1 \qquad \forall t \tag{14}$$

as can be readily derived from the completeness relation $\frac{1}{2}\sum_{\mu=0}^{4}(\sigma_\mu)_{ab}(\sigma_\mu)_{cd} = \delta_{ad}\delta_{bc}$, with $\sigma_0 = 1$.

The density matrix of a two-level system (like the one considered above) can always be expressed in terms of the Bloch vector (11), according to the formula

$$\rho = \frac{1}{2}(\mathbf{1} + \mathbf{x}\cdot\boldsymbol{\sigma}) \tag{15}$$

Notice that ρ is normalized $[\mathrm{Tr}(\rho) = 1]$ and $\mathrm{Tr}(\rho\boldsymbol{\sigma}) = \mathbf{x}$. Pure states are characterized by $\|\mathbf{x}\| = 1$.

The Lindblad equation (8) for the BBCDS model reads

$$\frac{d}{dt}\bar{\rho} = -i[\alpha\sigma_1, \bar{\rho}] - \beta^2(\bar{\rho} - \sigma_3\bar{\rho}\sigma_3) \tag{16}$$

This equation holds for the *averaged* density matrix. By making use of the explicit expression (15), one obtains

$$\begin{aligned}
\frac{d}{dt}\bar{x} &= -2\beta^2\bar{x}\\
\frac{d}{dt}\bar{y} &= -2\alpha\bar{z} - 2\beta^2\bar{y}\\
\frac{d}{dt}\bar{z} &= 2\alpha\bar{y}
\end{aligned} \tag{17}$$

which yield the solutions

$$\begin{aligned}
\bar{x}(t) &= \bar{x}(0)e^{-2\beta^2 t}\\
\bar{y}(t) &= e^{-\beta^2 t}[\bar{y}(0)\cos\omega t + c_1 \sin\omega t]\\
\bar{z}(t) &= e^{-\beta^2 t}[\bar{z}(0)\cos\omega t + c_2 \sin\omega t]
\end{aligned} \tag{18}$$

where $c_1 = [-\beta^2\bar{y}(0) - 2\alpha\bar{z}(0)]/\omega$ and $c_2 = [\beta^2\bar{z}(0) + 2\alpha\bar{y}(0)]/\omega$ and $\omega = \sqrt{4\alpha^2 - \beta^4}$ can be real or purely imaginary. Note that if $4\alpha^2 - \beta^4 < 0$ the solution is simply obtained by replacing the trigonometric functions in (18) with the hyperbolic ones: $\cos\omega t \to \cosh\omega t$, $\sin\omega t \to \sinh\omega t$. All these results are obviously in agreement with those obtained by BBCDS [6] and Berry [7], by making use of different techniques.

3. THE DECOHERENCE PARAMETER

The vector $\mathbf{x}$ is a stochastic process, and is such that $\|\mathbf{x}(t)\| = 1$, $\forall$t. As a consequence, from Eq. (15) one easily derives

$$\rho^2(t) = \rho(t) \qquad \forall t \tag{19}$$

For every realization of the stochastic process, or, in other words, if the quantity $\eta(t)$ in (1) or (9) is given a meaning as an ordinary function of time, the system remains in a pure state. On the other hand, if we are only interested in the expectation values of the system operators, the relevant density matrix is $\bar{\rho}$ (i.e., the one averaged over η), which satisfies

$$\bar{\rho}^2(t) = \tfrac{1}{4}[\mathbf{1} + 2\bar{\mathbf{x}}(t)\cdot\boldsymbol{\sigma} + \bar{\mathbf{x}}^2(t)] \neq \overline{\rho^2} = \bar{\rho} \tag{20}$$

Introduce the parameter

$$\begin{aligned}
\varepsilon &\equiv 2\,\mathrm{Tr}(\Delta\rho) \equiv 2\,\mathrm{Tr}(\overline{\rho^2} - \bar{\rho}^2) = 2\,\mathrm{Tr}(\bar{\rho} - \bar{\rho}^2)\\
&= 1 - \bar{\mathbf{x}}^2(t)
\end{aligned} \tag{21}$$

Notice that $0 \le \varepsilon \le 1$. This quantity measures how far $\bar{\rho}$ is from being idempotent [10], yielding a quantitative estimate of the degree of decoherence of the quantum system: if $\varepsilon = 0$, coherence is preserved; if, on the other hand, $\varepsilon = 1$, coherence is totally lost and the system is totally incoherent. For this reason, this parameter was named the *decoherence parameter* [11]. It is a very useful quantity in the theory of quantum mechanical measurements [12]. It is noteworthy that all the coherence properties of our system can be summarized by a numerical variable like our ε. This is an interesting characteristic of the model considered.

It is easy to check from the explicit expression (18) that $\varepsilon \to 1$ exponentially as $t \to \infty$: the quantum coherence is lost as time goes by, as a consequence of the noise term in the Hamiltonian (9). Such a loss of coherence is irretrievable, as one easily understands from physical considerations as well as from the structure (8) of the dynamical evolution law.

4. HINDERED EVOLUTION AND QUANTUM ZENO EFFECT

The interest of the above model lies in the different dynamical regimes that are obtained by varying the strength β of the coupling with the environment: if β is small, the interaction with the environment is weak and the system undergoes coherent quantum oscillations between its two states. If, on the other hand, β is large, the oscillations are hindered and the system becomes "localized" in one of its two states. BBCDS called this phenomenon "localization stabilized by noise" [6], and Berry emphasized the links with the quantum Zeno effect [7].

The purpose of the present section is to shed some additional light on the latter regime and the quantum Zeno effect. To this end, it is convenient to analyze a particular case: prepare the system in the initial state $\overline{x}(0) = \overline{y}(0) = 0, \overline{z}(0) = 1$ (all particles in state $|\psi_+\rangle$). If the coupling with the environment is large $\beta^2 \gg 2\alpha$, the solution is

$$\overline{\mathbf{x}}(t) = e^{-\beta^2 t}\begin{pmatrix} 0 \\ -\dfrac{2\alpha}{\omega}\sinh \omega t \\ \cosh \omega t + \dfrac{\beta^2}{\omega}\sinh \omega t \end{pmatrix} \overset{\beta^2 \gg 2\alpha;\text{large } t}{\simeq} e^{-(2\alpha^2/\beta^2)t}\begin{pmatrix} 0 \\ O\left(\dfrac{\alpha}{\beta^2}\right) \\ 1 + O\left(\dfrac{\alpha^2}{\beta^4}\right) \end{pmatrix} \tag{22}$$

where "large t" means $t \gg \omega^{-1} \simeq \beta^{-2}$ (observe that β^{-2} is a small quantity, so the above expansion is valid for rather small t). As one can see, when β is large the oscillations are hindered and the system tends to remain in its initial state:

$$\overline{z}(t) \simeq e^{-(2\alpha^2/\beta^2)t} \xrightarrow{\beta\to\infty} 1 \tag{23}$$

for $t < \infty$. This "halting" of the quantum evolution due to strong coupling with the environment is familiar in a variety of physical situations: a remarkable and well-known example is the explanation of the chiral nature of certain molecules [13].

Let us now take a different approach. Assume that the coupling with the environment is small $\beta^2 \ll 2\alpha$, but *frequent* measurements are performed on the system in order to ascertain whether it is localized in one of the eigenstates of σ_3: $|\psi_+\rangle$ or $|\psi_-\rangle$. This is the usual framework of "pulsed" observation, typical of the quantum Zeno effect [8].

Since we are considering small coupling ($\beta^2 \ll 2\alpha$), the solution of the stochastic differential equation is

$$\overline{\mathbf{x}}(t) = e^{-\beta^2 t}\begin{pmatrix} 0 \\ -\dfrac{2\alpha}{\omega}\sin \omega t \\ \cos \omega t + \dfrac{\beta^2}{\omega}\sin \omega t \end{pmatrix} \overset{\beta^2 \ll 2\alpha;\text{small } t}{\simeq} \begin{pmatrix} 0 \\ -2\alpha t \\ 1 - 2\alpha^2 t^2 \end{pmatrix} \tag{24}$$

where "small t" means $t \ll \omega^{-1} \simeq 2\alpha$.

If a σ_3 measurement is performed at time $t = \delta t_1$, $\mathbf{x}(t)$ "collapses" into the state $(0, 0, \lambda)$ with probability $p_\lambda = [1 + \lambda\overline{z}(\delta t_1)]/2$, where $\lambda = \pm 1$ is the result of the measurement. Consequently, the density matrix after one measurement becomes

$$\rho \to \overline{\rho}^{(1)} = \frac{1}{2}[\mathbf{1} + \overline{z}(\delta t_1)\sigma_3] \tag{25}$$

Notice that we are considering the average $\overline{\cdots}$. After the measurement, the evolution starts anew, with the initial condition $\mathbf{x}(\delta t_1) = (0, 0, \lambda)$, where λ is *either* $+1$ *or* -1, each event occurring with probability p_λ. The density matrix reads

$$\rho(t) = \frac{1 + \overline{z}(\delta t_1)}{2}\rho^{(+)}(t - \delta t_1) + \frac{1 - \overline{z}(\delta t_1)}{2}\rho^{(-)}(t - \delta t_1) \tag{26}$$

where $t - \delta t_1 > 0$ is the time elapsed after the measurement and

$$\rho^{(\lambda)}(\tau) = \frac{1}{2}[\mathbf{1} + \mathbf{x}(\tau; (0,0,\lambda)) \cdot \boldsymbol{\sigma}] \qquad (\lambda = \pm 1) \tag{27}$$

is the density matrix of the system after the measurement, $\mathbf{x}(\tau; (0, 0, \lambda))$ being the solution of the stochastic differential equation (12) with initial condition $(0, 0, \lambda)$. By (18), $\overline{\mathbf{x}}(\tau; (0, 0, \lambda)) = -\overline{\mathbf{x}}(\tau; (0, 0, -\lambda))$ and (26) yields, on the average,

$$\overline{\rho}(t) = \frac{1}{2}[\mathbf{1} + \overline{z}(\delta t_1)\overline{\mathbf{x}}(t - \delta t_1; (0, 0, 1)) \cdot \boldsymbol{\sigma}] \qquad (t > \delta t_1) \tag{28}$$

If another σ_3 measurement is performed at time $t = \delta t_1 + \delta t_2$,

$$\rho \to \overline{\rho}^{(2)} = \frac{1}{2}[\mathbf{1} + \overline{z}(\delta t_1)\overline{z}(\delta t_2)\sigma_3] \tag{29}$$

and the evolution starts again with the new initial conditions. If N measurements are performed in time $t = \Sigma_{j=1}^{N} \delta t_j$, the final state is

$$\overline{\rho}^{(N)} = \frac{1}{2}\left[\mathbf{1} + \prod_{j=1}^{N} \overline{z}(\delta t_j)\sigma_3\right] \tag{30}$$

Take, for simplicity, $\delta t_j = \delta t$ $(\forall j)$, so that the time interval between successive measurements is constant. Then, from (24),

$$\overline{z}(t) = \mathrm{Tr}(\overline{\rho}^{(N)}\sigma_3) = [\overline{z}(\delta t = t/N)]^N \xrightarrow{N\to\infty} e^{-(2\alpha^2\delta t)t} \xrightarrow{\delta t \to 0} 1, \tag{31}$$

for $t < \infty$. Once again, the oscillations are hindered. One can say that the two situations analyzed in this section, large coupling with the environment (23) and frequent measurements (31), are equivalent in that they yield the same physical effect. There is an analogy with what Schulman calls "continuous versus pulsed observations" [14] and a link with the phenomenon of "dominated evolution," analyzed in [15]. The two regimes can be quantitatively compared: if

$$\beta^{-2} = \delta t, \tag{32}$$

(23) and (31) are identical. In other words, a (σ_3) white noise of large strength β and a series of frequent (σ_3) observations at short time intervals δt slow down (and eventually halt) the evolution of an *eigenstate* of σ_3 [initial condition $\overline{z}(0) = 1$].

It is worth estimating the decoherence parameter ε introduced in the previous section, which measures the degree of quantum coherence lost during the interaction with the environment. Recall that the decoherence parameter ε is essentially given by the Bloch vector $\overline{\mathbf{x}}(t)$. See (21). Since $1 = \overline{\|\mathbf{x}(t)\|^2} \geq \overline{\mathbf{x}}^2(t) \geq \overline{z}^2(t)$ and $\overline{z}(t) \to 1$ in both limits considered above,

$$\varepsilon \to 0 \tag{33}$$

in both cases. This implies, in particular, in the latter case of frequent *measurements,* that the system remains in a pure state in the $N = \infty$ limit, as if no measurements were performed on it. This observation is in line with the claim [16] that von Neumann's projection postulate is not necessary to realize the quantum Zeno phenomenon.

5. ZENO DYNAMICS AND (DE)COHERENCE

The previous discussion is valid when the system is prepared in the particular initial state $\overline{x}(0) = \overline{y}(0) = 0, \overline{z}(0) = 1$, that is, in an eigenstate of σ_3. Since our Hamiltonian is given by (9), the system undergoes coherent (Rabi) oscillations around the first axis, modulated by the stochastic vibrating force along the third axis. This is why the system tends to remain in the initial state when the stochastic force becomes infinitely strong ($\beta \to \infty$), as in (23). In the previous section, this phenomenon of hindered transition was compared with the Zeno dynamics, i.e., the short-time characteristics of quantum systems.

The question we wish to investigate here is whether such a similarity between the effect of the strong environmental force, here modeled by the stochastic term in the Hamiltonian, and the Zeno dynamics can be given a more general meaning. For this purpose, we consider a system characterized by the same Hamiltonian H (9), but prepared in different initial states.

If the system is prepared in an eigenstate of σ_1, i.e., $\overline{x}(0) = 1$, its dynamics is trivial, as one easily understands either from the structure of H or from the solutions (18):

$$\overline{x}(t) = e^{-2\beta^2 t} \qquad \overline{y}(t) = \overline{z}(t) = 0 \tag{34}$$

The system decays into a completely mixed state in the large-t limit, which is a consequence of the fact that the decoherence parameter ε approaches 1:

$$\varepsilon = 1 - \overline{\mathbf{x}}^2(t) \xrightarrow{t\to\infty} 1. \tag{35}$$

If, on the other hand, the same system is frequently monitored to check whether it remains in the initial state, an argument similar to that in the previous section leads to the final density matrix,

$$\overline{\rho}^{(N)} = \frac{1}{2}[\mathbf{1} + [\overline{x}(t/N)]^N\sigma_1] \tag{36}$$

when N σ_1 measurements are performed at equal time intervals within the finite time t. Notice that $\overline{x}(t)$ is just the exponential function (34) and therefore the above quantity is independent of N:

$$\overline{\rho}^{(N)} = \frac{1}{2}[\mathbf{1} + e^{-2\beta^2 t}\sigma_1] \tag{37}$$

This result was to be expected, since the Hamiltonian is nothing but a fluctuating force, for a system prepared in one of the eigenstates of σ_1, so that the system decays exponentially at all times, regardless of the frequency of monitoring. In conclusion, the system exponentially loses quantum coherence:

$$\varepsilon = 1 - e^{-4\beta^2 t} \xrightarrow{t\to\infty} 1 \tag{38}$$

As a final simple example, prepare the system in the state $\bar{y}(0) = 1$. This case represents an interesting situation, since the system undergoes both coherent and stochastic forces from the beginning. If $\beta^4 < 2\alpha^2$, the solutions is

$$\begin{aligned}\bar{x}(t) &= 0\\ \bar{y}(t) &= e^{-\beta^2 t}[\cos\omega t - (\beta^2/\omega)\sin\omega t]\\ \bar{z}(t) &= (2\alpha/\omega)e^{-\beta^2 t}\sin\omega t\end{aligned} \tag{39}$$

If, on the other hand, $\beta^4 > 2\alpha^2$, the solution is

$$\bar{\mathbf{x}}(t) = e^{-\beta^2 t}\begin{pmatrix} 0 \\ \cosh\omega t - (\beta^2/\omega)\sinh\omega t \\ (2\alpha/\omega)\sinh\omega t\end{pmatrix} \tag{40}$$

which reduces for large $t \gg \omega^{-1} \simeq \beta^{-2}$, to

$$\bar{\mathbf{x}}(t) \xrightarrow{\text{large}\, t} e^{-(2\alpha^2/\beta^2)t}\begin{pmatrix} 0 \\ -\alpha^2/\beta^4 \\ \alpha/\beta^2\end{pmatrix} \tag{41}$$

Notice that unlike the case considered in the previous section (22)–(23), in this example even a very strong noise ($\beta \to \infty$) does not yield a pure state. Instead, the system finally loses quantum coherence and the decoherence parameter tends to 1 as a power function of β:

$$\varepsilon \overset{\text{large}\,\beta}{\simeq} 1 - (\alpha^2/\beta^4)e^{-4(\alpha^2/\beta^2)t} \tag{42}$$

What happens if the system prepared in such initial state is frequently monitored, by measuring σ_2? The answer is easily inferred from the final density matrix,

$$\bar{\rho}^{(N)} = \frac{1}{2}[\mathbf{1} + [\bar{y}(t/N)]^N\sigma_2] \tag{43}$$

which is easily derived like (36). Since (40) yields, for small t,

$$\bar{y}(t) \simeq 1 - 2\beta^2 t \tag{44}$$

the above density matrix becomes again independent of N,

$$\bar{\rho}^{(N)} \sim \frac{1}{2}[\mathbf{1} + e^{-2\beta^2 t}\sigma_2] \tag{45}$$

and the decoherence parameter is given again by (38).

In both cases considered in this section, neither a strong coupling with the environment, nor a frequent series of measurements can halt the evolution. There is no quantum Zeno effect.

6. SUMMARY AND COMMENTS

We have studied the model proposed by Blanchard, Bolz, Cini, De Angelis, and Serva [6], showing its explicit solution and discussing the loss of quantum mechanical coherence. The coherence properties of the system are well described by a single parameter ε, which is defined in terms of the (averaged) Bloch vector $\bar{\mathbf{x}}(t)$. Coherence is lost exponentially as $t \to \infty$. It is remarkable that ε, which can be evaluated from $\bar{\mathbf{x}}(t)$, yields a quantitative estimate of the quantity $\bar{\rho} - \bar{\rho}^2$, even though the term $\bar{\rho}^2$ is sometimes claimed not to be directly measurable.

We have concentrated our attention, in particular, on the short-time dynamics of the system. With the Hamiltonian (9), if the system is prepared in a particular state, i.e., one of the σ_3 eigenstates, the quantum coherence can be preserved by increasing either the strength of the noise term ($\beta \to \infty$) or the frequency of σ_3 observations. The latter case is a realization of the quantum Zeno phenomenon. For different initial conditions and different types of measurements, the system generally loses its quantum coherence due to the action of the stochastic force. The degree of the loss of coherence is manifest in the value of the decoherence parameter ε; since the latter behaves exponentially in such cases, *no* Zeno effect can be obtained even under very frequent observations.

In conclusion, the occurrence of the Zeno phenomenon in the present context requires the concomitance of several factors: in particular, the noise term (or a series of frequent observations) prevents the system from decohering if and only if it is applied in an appropriate direction with respect to the initial state of the system.

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Quantum stochastic resonance in driven spin-boson system with stochastic limit approximation

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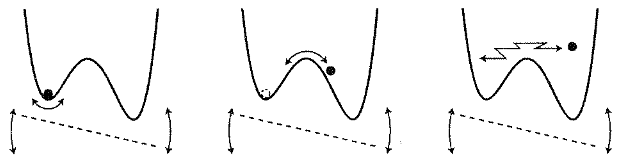
After a brief review of stochastic limit approximation with spin-boson system from physical points of view, amplification phenomenon—stochastic resonance phenomenon—in driven spin-boson system is observed which is helped by the quantum white noise introduced through the stochastic limit approximation. Signal-to-noise ratio resonates at certain temperature if another noise parameter η is chosen properly. Not only the stochastic resonance in usual sense, but also the possibilities of the new and interesting phenomena—"anti-resonance" and "double resonance"—are shown with some choices of η. The shift in frequency of the system due to the interaction with the environment—Lamb shift—has an important role in these phenomena.

I. INTRODUCTION

Stochastic resonance (SR) phenomena were first discovered in connection with periodically recurrent glacial age. Since then this phenomenon has been found to occur in various fields and has been attracting wide attention. In short SR is phenomenon whereby, in contrast to common sense, added noise seems to help to amplify a signal. Let us briefly review SR phenomenon using the bistable potential model, driven by a periodic perturbation. A classical particle in a potential $V(x)$, which has two local minima, is perturbed by a periodic external force with an amplitude ξ and a frequency Ω under the influence of noise (Fig. 1). If the amplitude ξ is small, the particle in one of the stable states cannot go over the potential barrier to the other stable state [Fig. 1(a)], in other words, the system does not respond to the input perturbation. The addition of noise changes the situation; now the particle is kicked by the random force and can go over the barrier. However if the noise is too strong the response to the input signal may be smeared. This is because in this case the particle moves randomly irrespective of the periodicity of the perturbation [Fig. 1(c)]. However at a certain added noise strength the particle can be made to travel back and forth between the two stable state, *synchronizing with periodic perturbation of frequency* Ω [Fig. 1(b)]. That is the system responds to the input.



(a) (b) (c)

FIG. 1. Response of a particle in a bistable potential to an external periodic perturbation with (a) too small, (b) appropriate, and (c) too strong noise.

More precisely we can characterize SR as follow:

(1) The power spectrum of the response of a system to a periodic input has a main sharp peak at the input frequency Ω *if the noise strength (or temperature) is chosen properly.*

(2) The signal-to-noise ration (SNR) of the response resonates *at a certain noise strength (or at a certain temperature).*

Besides the periodicity in the emergence of glacial ages [2], SR phenomenon are widely found in nature. For example SR is found in the nerve of the flagellum of a crayfish's tail [3] (however in this case, unlike the bistable system described above, a threshold reaction is triggered by noise). Therefore SR may be a universal concept.

In this paper we discuss SR in bistable model *at the quantum level*, that is quantum stochastic resonance (QSR). Of course this effect has already been widely studied [1,4], however our particular interest is "quantum noise," or "quantum dynamics with dissipation." That is, we are interested in how noise is introduced into the quantum dynamics to produce QSR. This is not only important question for QSR, but it is also relevant for the understanding of several other fundamental aspects of quantum mechanics, that is the problems of relaxation,

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decoherence, measurement and so on. However quantum mechanics is usually written in terms of causal deterministic theory governed by unitary time evolution. It is therefore hard, in principle, to introduce the notion of "noise" or "dissipation" (with finite degrees of freedom).

In these circumstances there are several different ways to proceed. One of the most popular approaches is to introduce an "environment," "reservoir" or "heat bath," whose detailed specification one does not know but which has infinite degrees of freedom. The whole system (*i.e.* "system" + "environment") is then treated in the quantum mechanical way [5–8].

Through this interaction, the system exchanges energy with the environment—"dissipation"—, and then it is reasonable to assume that some kind of "noise" or "fluctuation" would appear due to some "fluctuation-dissipation relation."

Along these lines Accardi *et al.* [9–11] have introduced the stochastic limit approximation (SLA) as a way to realize "quantum white noise." The SLA is one way to deal with the van Hove limit, which is the weak coupling limit given by, $\lambda \to 0$ and time coarse-graining limit given by, $t \mapsto \tau = \lambda^2 t$. This limit ensures that a system in a heat bath approaches canonical state [13].

As is explicitly shown in Sec. III for the spin-boson system, the spin system in the heat bath composed of bosons actually approaches the canonical state under the SLA. Furthermore, one can discuss important properties in quantum dissipative dynamics within this framework, such as the dependence on temperature of the shift in frequency of the system due to the interaction with the heat bath.

We here focus our attention on the quantum white noise introduced through the SLA, and study QSR as part of investigations of the properties of this noise. After the introduction of the model to be studied—the driven spin-boson system—in Sec. II, the SLA is briefly reviewed in Sec. III with the spin-boson system. Using this method, we discuss, in Sec. IV, QSR in the driven spin-boson system and the role of the quantum fluctuation and dissipation introduced through the SLA. Section V is devoted to concluding remarks with comments on the experimental feasibility of the phenomenon studied here. In the Appendix, we add comments on the SLA from a physical point of view.

II. DRIVEN SPIN-BOSON SYSTEM

Here let us introduce the model—the driven spin-boson system [1,4]—as a special case of the bistable model in Sec. I.

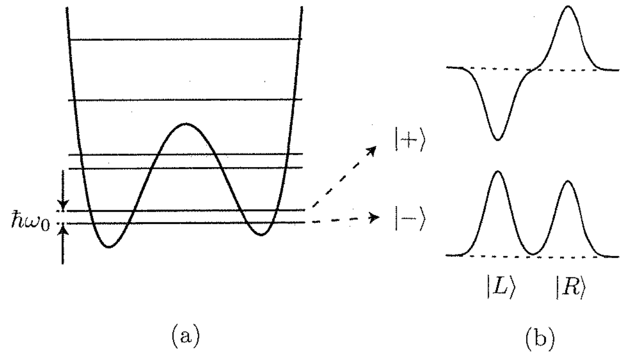

FIG. 2. Bistable system at the quantum scale.

A. Spin system

Consider the situation where the system illustrated in Fig. 2 is in a deep cold heat bath and its dynamics are ruled mainly by the lowest tunnel-splitted pair of levels $|\pm\rangle$, where thermal hopping to upper levels can be neglected. If the tunneling amplitude between the two wells is sufficiently small, we are able to consider two "localized" states, $|L\rangle$ and $|R\rangle$, which are approximately the ground states of left and right wells, respectively. Taking the set of these states as the Hilbert space basis, this system can be described by the Hamiltonian,

$$H_S = \frac{\epsilon}{2}\Big(|R\rangle\langle R| - |L\rangle\langle L|\Big) + \frac{\Delta}{2}\Big(|R\rangle\langle L| + |L\rangle\langle R|\Big), \tag{2.1}$$

which is essentially the spin-1/2 Hamiltonian with the parameter Δ characterizing the tunneling amplitude between the two wells, and ϵ characterizing the difference in energy between the $|L\rangle$ and $|R\rangle$ states. Hereafter, we call it the spin system. By introducing a new basis rotated by the angle $\theta = \cos^{-1}(\epsilon/\omega_0) = \sin^{-1}(\Delta/\omega_0)$,

$$|+\rangle = \cos\frac{\theta}{2}|R\rangle + \sin\frac{\theta}{2}|L\rangle, \tag{2.2a}$$

$$|-\rangle = -\sin\frac{\theta}{2}|R\rangle + \cos\frac{\theta}{2}|L\rangle, \tag{2.2b}$$

the Hamiltonian H_S is rewritten as a diagonal form

$$H_S = \frac{\omega_0}{2}\Big(|+\rangle\langle+| - |-\rangle\langle-|\Big). \tag{2.3}$$

The energy gap ω_0 between the two lowest states, $|+\rangle$ and $|-\rangle$, is given by

$$\omega_0 = \sqrt{\epsilon^2 + \Delta^2}. \tag{2.4}$$

This two-level system is driven by a periodic forcing with frequency Ω and amplitude ξ. This applied force can be described by the perturbative Hamiltonian

$$W = \xi X \sin \Omega t \tag{2.5}$$

with the "position" operator X defined by

$$X = |R\rangle\langle R| - |L\rangle\langle L|. \tag{2.6}$$

Of course, there are many other possibilities for the system driving instead of (2.5), e.g., $W' = \xi(|R\rangle\langle L| + |L\rangle\langle R|) \sin \Omega t$, but we choose the perturbation (2.5) since it corresponds to classical SR in the bistable model.

Note that X is an order parameter in discussing QSR phenomenon in Sec. IV, which measures the transitions between the states $|L\rangle$ and $|R\rangle$ under the influence of the external perturbation.

B. Boson system and its interaction with the spin system

As mentioned in Sec. I, one must introduce an "environment" for the spin system to dissipate and be disturbed. The environment is chosen as a set of bosons in this paper, whose Hamiltonian is given by

$$H_B = \int dk\, \omega_k a_k^\dagger a_k. \tag{2.7}$$

Here, a_k and $a_k^\dagger$ are respectively annihilation and creation operators for a boson of mode k with energy $\omega_k > 0$, and satisfy the commutation relations

$$[a_k, a_{k'}^\dagger] = \delta(k - k'), \quad (\text{others}) = 0. \tag{2.8}$$

The spin system interacts with the bosons via the interaction Hamiltonian

$$\lambda V = \lambda X \int dk\, (g_k a_k^\dagger + g_k^* a_k), \tag{2.9}$$

where λ characterizes the strength of the interaction, and the structure function g_k is a coupling of the bosons of mode k with the spin system subject to the condition $\int dk\, |g_k|^2 < \infty$. Note that the spin system and the boson system are coupled with the bilinear product of the spin operator X and the boson operators. Although some specified choices of the coupling may result in certain outputs, the details of the microscopic Hamiltonian are not so significant for the derivation of damping dynamics. A comment on this point can be found in Sec. III B 2.

The system to be analyzed in this paper is thus given by the total Hamiltonian

$$H = H_0 + W + \lambda V, \quad H_0 = H_S + H_B. \tag{2.10}$$

III. STOCHASTIC LIMIT APPROXIMATION

In this section we briefly review the stochastic limit approximation (SLA) formulated by Accardi *et al.* [9–11]. For simplicity, let us consider the case where there is no external perturbation, i.e., $\xi = 0$ [10,11,7]. The Hamiltonian of the system concerned in this section is thus

$$H_{\text{SB}} = H_0 + \lambda V. \tag{3.1}$$

We entrust the mathematical details to Ref. [9] or [11], but note that several physically important points are emphasized and added to the work in Ref. [10] and [11]. Furthermore in the appendix we add some comments on the SLA taken from slightly different point of view to that taken by Accardi *et al.*

A. Application to spin-boson system

In the interaction picture, the time-evolution operator $U_I^{(\lambda)}(t)$ which is governed by the Hamiltonian (3.1) satisfies the Tomonaga–Schwinger equation

$$\frac{d}{dt} U_I^{(\lambda)}(t) = -i\lambda V_I(t) U_I^{(\lambda)}(t), \quad U_I^{(\lambda)}(0) = 1, \tag{3.2a}$$

$$V_I(t) = e^{iH_0 t} V e^{-iH_0 t}, \tag{3.2b}$$

or specifically

$$\frac{d}{dt} U_I^{(\lambda)}(t) = -i\lambda \sum_\alpha \left(D_\alpha A_\alpha^\dagger(t) + D_\alpha^\dagger A_\alpha(t) \right) U_I^{(\lambda)}(t), \tag{3.3}$$

where α takes $\alpha = \{+, -, 0\}$,

$$D_\pm = |\pm\rangle\langle\mp|, \quad D_0 = |+\rangle\langle+| - |-\rangle\langle-| \tag{3.4}$$

are the spin system operators, and

$$A_\pm(t) = -\frac{\Delta}{\omega_0} \int dk\, g_k^* a_k e^{-i(\omega_k \pm \omega_0)t}, \tag{3.5a}$$

$$A_0(t) = \frac{\epsilon}{\omega_0} \int dk\, g_k^* a_k e^{-i\omega_k t} \tag{3.5b}$$

are the boson system operators. The SLA is prescribed in the Tomonaga–Schwinger equation (3.3) by rescaling time as $t \mapsto \tau = \lambda^2 t$,

$$\frac{d}{d\tau} U_I^{(\lambda)}(\tau/\lambda^2) = -i\frac{1}{\lambda} \sum_\alpha \left(D_\alpha A_\alpha^\dagger(\tau/\lambda^2) + D_\alpha^\dagger A_\alpha(\tau/\lambda^2) \right) \times U_I^{(\lambda)}(\tau/\lambda^2), \tag{3.6}$$

and then the weak coupling limit $\lambda \to 0$ is taken (i.e., the van Hove limit [12,13]). As proved in Ref. [9] or [11], there exist the limits

$$\frac{1}{\lambda}A_\alpha(\tau/\lambda^2) \to b_\alpha(\tau), \quad \frac{1}{\lambda}A_\alpha^\dagger(\tau/\lambda^2) \to b_\alpha^\dagger(\tau), \tag{3.7}$$

$$U_I^{(\lambda)}(\tau/\lambda^2) \to U_I(\tau), \tag{3.8}$$

and formally

$$\frac{d}{d\tau}U_I(\tau) = -i\sum_\alpha \Big(D_\alpha b_\alpha^\dagger(\tau) + D_\alpha^\dagger b_\alpha(\tau)\Big)\,U_I(\tau). \tag{3.9}$$

In this limit, the boson operators $b_\alpha(\tau)$ and $b_\alpha^\dagger(\tau)$ satisfy the commutation relations [10,11]

$$[b_-(\tau), b_-^\dagger(\tau')] = 2\gamma\delta(\tau-\tau'), \quad (\text{others}) = 0 \tag{3.10}$$

with

$$\gamma = \left(\frac{\Delta}{\omega_0}\right)^2 J(\omega_0), \tag{3.11}$$

$$J(\omega) = \pi\int dk\,|g_k|^2\delta(\omega_k - \omega). \tag{3.12}$$

For comments on these limits from a physical point of view, see the Appendix. The commutation relations (3.10) allow one reasonably to call $b_\alpha(\tau)$ and $b_\alpha^\dagger(\tau)$ "quantum white noise," and the Tomonaga–Schwinger equation (3.9) the "quantum Langevin equation." The correlation time is vanishingly small. However at the same time, we should be careful to note that equation (3.9) is ill-defined. Fortunately, however, noticing the commutators [10,11]

$$[b_\pm(\tau), U_I(\tau)] = -i\left(\frac{\Delta}{\omega_0}\right)^2\gamma(\mp\omega_0)D_\pm U_I(\tau), \tag{3.13a}$$

$$[b_0(\tau), U_I(\tau)] = -i\left(\frac{\epsilon}{\omega_0}\right)^2\gamma(0)D_0U_I(\tau) \tag{3.13b}$$

with

$$\gamma(\omega) = J(\omega) - iI(\omega), \quad I(\omega) = \frac{1}{\pi}\mathcal{P}\int d\omega'\,\frac{J(\omega')}{\omega'-\omega}, \tag{3.14}$$

one can evaluate the evolutions of some physically important quantities. For example, for the special initial state density operator

$$\rho = \rho_S\otimes\rho_B, \quad \rho_B = |0\rangle\langle 0| \tag{3.15}$$

(i.e., the spin system and the boson system are uncorrelated and the boson system is in the ground state at $\tau = 0$), the equations for the spin system operators defined by

$$D_\alpha(\tau) = \mathrm{tr}_B\Big(\rho_B\,e^{iH_{\mathrm{SB}}\tau/\lambda^2}D_\alpha e^{-iH_{\mathrm{SB}}\tau/\lambda^2}\Big) \tag{3.16}$$

can be obtained as

$$\frac{d}{d\tau}D_\pm(\tau) = -(\gamma\mp i\omega_R)D_\pm(\tau), \tag{3.17a}$$

$$\frac{d}{d\tau}D_0(\tau) = -2\gamma D_0(\tau) - 2\gamma, \tag{3.17b}$$

which give the exponentially decaying dynamics

$$D_\pm(\tau) = D_\pm e^{-(\gamma\mp i\omega_R)\tau}, \tag{3.18a}$$

$$D_0(\tau) = (D_0+1)e^{-2\gamma\tau} - 1. \tag{3.18b}$$

Here ω_R is the renormalized frequency

$$\omega_R = \omega_0/\lambda^2 - \sigma = \tilde{\omega}_0 - \sigma, \tag{3.19}$$

where the frequency shift σ emerges due to the interaction

$$\sigma = \left(\frac{\Delta}{\omega_0}\right)^2\Big(I(\omega_0) - I(-\omega_0)\Big). \tag{3.20}$$

Note that tr_B denotes the trace over the boson-degrees of freedom. This is the procedure for "partial trace." It reduces the effects of the interaction between the spin system and the boson environment to the spectral function $J(\omega)$ defined in Eq. (3.12). The damping coefficient γ in Eq. (3.11) and the frequency shift σ in Eq. (3.20) with Eq. (3.14) are both given in terms of $J(\omega)$.

It is also possible to evaluate γ and σ for the boson environment at finite temperature T

$$\rho_B = e^{-\beta H_B}/\,\mathrm{tr}_B\,e^{-\beta H_B} \tag{3.21}$$

by using the TFD technique [14], for example. Here $\beta = 1/k_BT$ with k_B being the Boltzmann constant. In this case, one obtains

$$D_\pm(\tau) = D_\pm e^{-(\gamma^\beta\mp i\omega_R^\beta)\tau}, \tag{3.22a}$$

$$D_0(\tau) = \left(D_0 + \frac{\gamma}{\gamma^\beta}\right)e^{-2\gamma^\beta\tau} - \frac{\gamma}{\gamma^\beta} \tag{3.22b}$$

with the temperature affected parameters

$$\gamma^\beta = \left(\frac{\Delta}{\omega_0}\right)^2 J^\beta(\omega_0), \tag{3.23a}$$

$$\omega_R^\beta = \omega_0/\lambda^2 - \sigma^\beta = \tilde{\omega}_0 - \sigma^\beta, \tag{3.23b}$$

$$\sigma^\beta = \left(\frac{\Delta}{\omega_0}\right)^2\Big(I^\beta(\omega_0) - I^\beta(-\omega_0)\Big), \tag{3.23c}$$

and the functions

$$J^\beta(\omega) = J(\omega)\coth\frac{1}{2}\beta\omega, \tag{3.24a}$$

$$I^{\beta}(\omega) = \frac{1}{\pi}\mathcal{P}\int d\omega' \, \frac{J^{\beta}(\omega')}{\omega' - \omega}. \tag{3.24b}$$

The damping coefficient γ^{β} and the frequency shift σ^{β} are obtained from γ and σ, respectively, by replacing the spectral function $J(\omega)$ with the temperature modified one $J^{\beta}(\omega)$.

Notice that the long-time limits of the operators $D_0(\tau) \to -\tanh(\beta\omega_0/2)$ and $D_{\pm}(\tau) \to 0$ are both c-numbers (or unit operators of the spin system multiplied by c-numbers). This means that the spin system approaches some unique state irrespective of the initial state ρ_S. In fact, the averages of any spin system operators, which are composed of $D_{\alpha}(\tau)$, approach unique values. One can further confirm that the long-time limit of the state of the spin system is nothing but the thermal state at the temperature T. Taking averages of $D_{\alpha}(\tau)$ with some arbitrary initial state ρ_S, one obtains the matrix elements of the system density operator defined by

$$\rho_S(\tau) = \mathrm{tr}_B \, \rho(\tau), \tag{3.25}$$

$$\rho(\tau) = e^{-iH_{\mathrm{SB}}\tau/\lambda^2} \rho \, e^{iH_{\mathrm{SB}}\tau/\lambda^2}. \tag{3.26}$$

Their dynamics are immediately obtained from the equations (3.22), and their long-time limits are given by

$$\langle -|\rho_S(\tau)|+\rangle = \langle D_+(\tau)\rangle \to 0, \tag{3.27a}$$

$$\langle \pm|\rho_S(\tau)|\pm\rangle = \frac{1}{2}\Big(1 \pm \langle D_0(\tau)\rangle\Big) \to \frac{e^{\mp\beta\omega_0/2}}{e^{\beta\omega_0/2} + e^{-\beta\omega_0/2}}, \tag{3.27b}$$

which are equivalent to

$$\rho_S(\tau) \to e^{-\beta H_S} / \mathrm{tr}_S \, e^{-\beta H_S}, \tag{3.28}$$

i.e., the system approaches the thermal equilibrium state at temperature T through decoherence (3.27a). Here tr_S denotes the trace over the spin-degrees of freedom.

B. Comments from physical points of view

1. Orders of parameters

It is important to clarify the order of magnitude of the parameters. In this formalism, one considers that the new time τ is physical and that, if they are measured in this macroscopic time, the parameters of the spin system should have some meaningful values, e.g., ω_R^{β} or $\tilde{\omega}_0$, instead of ω_0, should be finite. On the other hand, the time scales of the boson system should be measured in the microscopic time t. This can be seen in the emergence of the delta function in Eq. (3.10). This is due to the coarse-graining in time, $t \mapsto \tau = \lambda^2 t \; (\lambda \to 0)$. It reflects the fact that characteristic time scales of the boson system, like the correlation time for example, are vanishingly small when measured in the macroscopic time. That is they are negligible when compared to the characteristic times of the spin system, such as $1/\tilde{\omega}_0$ for example. This is the situation which occurs in the stochastic limit.

As for the temperature, the physically interesting situation is where the temperature T is such that $\beta\omega_0$ has some finite value, which is contained in γ^{β} and σ^{β} through $J^{\beta}(\omega_0)$ [see Eqs. (3.23) and (3.24)] and in the thermal equilibrium distribution (3.27b). That is, the rescaled temperature $\tilde{T} = T/\lambda^2$ should be finite.

2. Choice of the coupling

In this paper, the spin-boson system is coupled through the specific choice of the coupling, in particular, the choice X. Of course, there are many other possibilities, such as $V = (|R\rangle\langle L| + |L\rangle\langle R|) \int dk \, (g_k a_k^{\dagger} + g_k^* a_k)$, but they all result in the same damping dynamics (3.22) except for the prefactors $(\Delta/\omega_0)^2$ and $(\epsilon/\omega_0)^2$ in the definitions of γ^{β} and σ^{β} in Eqs. (3.23a) and (3.23c). [One has to notice, however, that the special choice of the interaction $V = (|+\rangle\langle +| - |-\rangle\langle -|) \int dk \, (g_k a_k^{\dagger} + g_k^* a_k)$ gives the special situation $\gamma^{\beta} = 0$, which is also included in Eqs. (3.22). The possibility of $\sigma^{\beta} = 0$ also exists for some special choices of V.] From a semi-phenomenological point of view, Eqs.(3.22) are sufficient for the description of experiments. Knowledge of the parameters $\tilde{\omega}_0$, γ^{β}, and σ^{β} from an experiment would enable one to predict the dynamical development of any quantities. Details of the microscopic Hamiltonian would not have much effect on the macroscopic behavior.

IV. QUANTUM STOCHASTIC RESONANCE

Now let us discuss QSR in the driven spin-boson system (2.10) with the SLA. By observing the response of the system to the external perturbation (2.5) through the dynamics of the "position" operator

$$X(\tau) = \mathrm{tr}_B\Big(\rho_B \, e^{iH\tau/\lambda^2} X e^{-iH\tau/\lambda^2}\Big), \tag{4.1}$$

which measures the transitions of the system between the left state $|L\rangle$ and the right state $|R\rangle$, we see an amplification of the input external perturbation with the addition of noise.

The Tomonaga–Schwinger equation is now

$$\frac{d}{dt}U_I^{(\lambda)}(t) = -i\Big(\lambda V_I(t) + W_I(t)\Big)\, U_I^{(\lambda)}(t), \quad U_I^{(\lambda)}(0) = 1, \tag{4.2a}$$

$$W_I(t) = \xi \left[\frac{\epsilon}{\omega_0} D_0 - \frac{\Delta}{\omega_0} \left(D_+ e^{i\omega_0 t} + D_- e^{-i\omega_0 t} \right) \right] \sin \Omega t, \tag{4.2b}$$

and along the same lines as in Sec. III, the SLA ($t \mapsto \tau = \lambda^2 t$, $\lambda \to 0$) is taken to give the "quantum Langevin equation"

$$\begin{aligned} \frac{d}{d\tau} U_I(\tau) &= -i \sum_\alpha \Big(D_\alpha b_\alpha^\dagger(\tau) + D_\alpha^\dagger b_\alpha(\tau) \Big) U_I(\tau) \\ &\quad - i\tilde{\xi} \left[\frac{\epsilon}{\omega_0} D_0 - \frac{\Delta}{\omega_0} \left(D_+ e^{i\tilde{\omega}_0 \tau} + D_- e^{-i\tilde{\omega}_0 \tau} \right) \right] \\ &\quad \times \sin(\tilde{\Omega}\tau) U_I(\tau). \end{aligned} \tag{4.3}$$

Note that the parameters are rescaled as $\tilde{\omega}_0 = \omega_0/\lambda^2$, $\tilde{\Omega} = \Omega/\lambda^2$, and $\tilde{\xi} = \xi/\lambda^2$ according to time rescaling, and are assumed to take physical values if measured in the macroscopic time. The equations of the spin system operators $D_\alpha(\tau)$ are then given by

$$\begin{aligned} \frac{d}{d\tau} D_\pm(\tau) &= -(\gamma^\beta \mp i\omega_R^\beta) D_\pm(\tau) \\ &\quad \pm 2i\tilde{\xi} \left(\frac{\epsilon}{\omega_0} D_\pm(\tau) + \frac{1}{2} \frac{\Delta}{\omega_0} D_0(\tau) \right) \sin \tilde{\Omega}\tau, \end{aligned} \tag{4.4a}$$

$$\begin{aligned} \frac{d}{d\tau} D_0(\tau) &= -\gamma^\beta D_0(\tau) - \gamma \\ &\quad + 2i\tilde{\xi} \frac{\Delta}{\omega_0} \Big(D_+(\tau) - D_-(\tau) \Big) \sin \tilde{\Omega}\tau. \end{aligned} \tag{4.4b}$$

These are, however, difficult to solve exactly, so we rely upon the perturbation method and assume that the external perturbation is weak. (One is interested here in SR phenomenon, i.e., amplification of *weak* inputs with the help of noise.) The solutions up to $O(\tilde{\xi}/\tilde{\omega}_0)$ or $O(\xi/\omega_0)$ are thus obtained for long times $\tau \gg 1/\gamma^\beta$ as

$$\begin{aligned} D_\pm(\tau) \to \mp \frac{i}{2} \tilde{\xi} \frac{\Delta}{\omega_0} \tanh \frac{1}{2} \beta\omega_0 & \left[\frac{1}{(\omega_R^\beta - \tilde{\Omega}) \pm i\gamma^\beta} e^{\pm i\tilde{\Omega}\tau} \right. \\ & \left. - \frac{1}{(\omega_R^\beta + \tilde{\Omega}) \pm i\gamma^\beta} e^{\mp i\tilde{\Omega}\tau} \right], \end{aligned} \tag{4.5a}$$

$$D_0(\tau) \to -\tanh \frac{1}{2} \beta\omega_0, \tag{4.5b}$$

and $X(\tau)$, by combining these solutions, as

$$\begin{aligned} X(\tau) &= \frac{\epsilon}{\omega_0} D_0(\tau) - \frac{\Delta}{\omega_0} \Big(D_+(\tau) + D_-(\tau) \Big) \\ &\to -\frac{\epsilon}{\omega_0} \tanh \frac{1}{2} \beta\omega_0 + \frac{\xi}{\omega_0} A^\beta(\tilde{\Omega}) \sin\Big(\tilde{\Omega}\tau - \phi^\beta(\tilde{\Omega}) \Big). \end{aligned} \tag{4.6}$$

Here the amplitude $A^\beta(\tilde{\Omega})$ and the phase delay $\phi^\beta(\tilde{\Omega})$ are given, respectively, by

$$A^\beta(\tilde{\Omega}) = \frac{2(\Delta/\omega_0)^2 \tilde{\omega}_0 \omega_R^\beta \tanh(\beta\omega_0/2)}{\sqrt{\left((\omega_R^\beta)^2 - \tilde{\Omega}^2 + (\gamma^\beta)^2 \right)^2 + \left(2\gamma^\beta \tilde{\Omega} \right)^2}} \tag{4.7}$$

and

$$\tan \phi^\beta(\tilde{\Omega}) = \frac{2\gamma^\beta \tilde{\Omega}}{(\omega_R^\beta)^2 - \tilde{\Omega}^2 + (\gamma^\beta)^2}. \tag{4.8}$$

Responding to the input perturbation, $X(\tau)$ oscillates around the thermal equilibrium state with the frequency of the perturbation $\tilde{\Omega}$.

Let us define the signal-to-noise ratio (SNR) $R^\beta(\tilde{\Omega})$ by

$$R^\beta(\tilde{\Omega}) = |A^\beta(\tilde{\Omega})| / (\gamma^\beta / \tilde{\omega}_0), \tag{4.9}$$

and study its dependence on the temperature. If this resonates at any certain temperature we conclude SR exists in this model. In the following, we show the analyses for two specific choices of the spectral function $J(\omega)$, as examples. One choice is

$$J(\omega) = \begin{cases} \eta\omega & (0 < \omega < \Lambda) \\ \eta\Lambda & (\omega > \Lambda) \end{cases}, \tag{4.10a}$$

called here the "Ohmic case" [Fig. 3(b)], and the other is

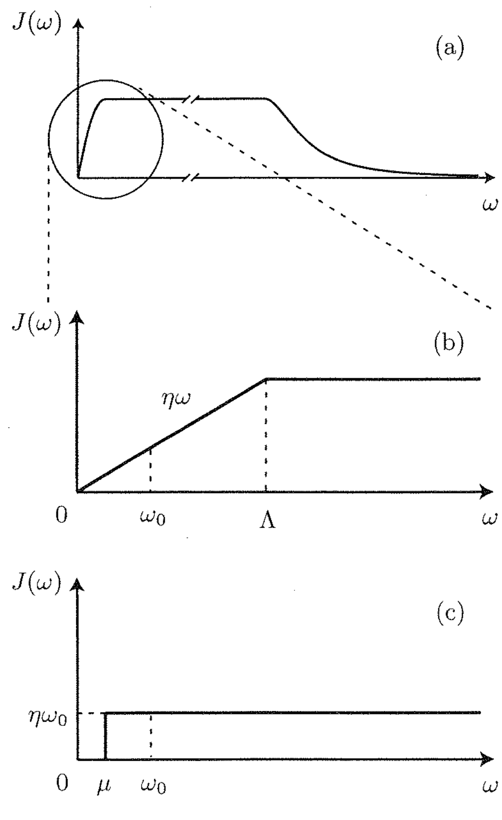

FIG. 3. Functional form of (a) physically realistic, (b) "Ohmic case," and (c) "constant case" spectral function $J(\omega)$.

$$J(\omega) = \begin{cases} 0 & (0 < \omega < \mu) \\ \eta\omega_0 & (\omega > \mu) \end{cases}, \tag{4.10b}$$

called the "constant case" [Fig. 3(c)]. Although a physically realistic spectral function $J(\omega)$ may have a cutoff at high frequency Λ_C as sketched in Fig. 3(a), it may be reasonable to consider that Λ_C can be infinitely large compared to the characteristic frequency of the spin system ω_0 in the stochastic limit situation. We hence adopt the model spectral functions given by Eqs. (4.10a) and (4.10b) and illustrated in Figs. 3(b) and (c). Their names come from the functional forms in the regions around "on-shell" $\omega = \omega_0$ which are assumed to be the regions $0 < \omega < \Lambda$ and $\mu < \omega$ for each case. Note that $J(\omega_0)$, and hence γ^β given by Eq. (3.23a), have the same value for both cases, and the dimensionless parameter η controls its magnitude, i.e., the noise strength. The difference between the two cases manifests itself in the temperature dependence of the frequency shift σ^β given by Eq. (3.23c) (Fig. 4). Note further that the function $I^\beta(\omega)$ defined by the dispersion relation (3.24b) does not converge with the model spectral functions (4.10). Since the asymptotic behaviors of $J^\beta(\omega)$ in these models are constant as $\omega \to \infty$, one has to apply a subtracted form to the dispersion relation. After a subtraction at $\omega = \omega_1$, this becomes

$$I^\beta(\omega) = I^\beta(\omega_1) + \frac{\omega - \omega_1}{\pi}\mathcal{P}\int d\omega' \frac{J^\beta(\omega') - J^\beta(\omega_1)}{(\omega' - \omega_1)(\omega' - \omega)}. \tag{4.11}$$

Choosing the subtraction point as $\omega_1 = 0$, one gets a convergent integral,

$$I^\beta(\omega) = I^\beta(0) + \frac{\omega}{\pi}\mathcal{P}\int d\omega' \frac{J^\beta(\omega')}{\omega'(\omega' - \omega)}. \tag{4.12}$$

From Eqs. (3.24b) and (4.12), the frequency shift is given by

$$\sigma^\beta = \left(\frac{\Delta}{\omega_0}\right)^2 \frac{2\omega_0}{\pi}\mathcal{P}\int d\omega' \frac{J^\beta(\omega')}{\omega'^2 - \omega_0^2}. \tag{4.13}$$

There are two important parameters concerning the environment or the noise, i.e., the temperature T and

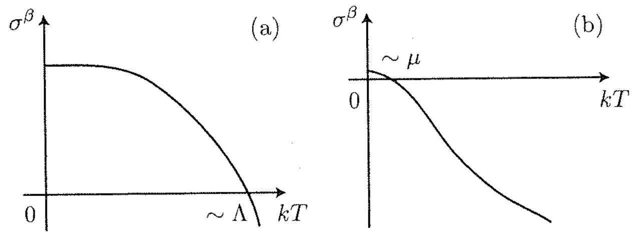

FIG. 4. Schematic forms of the frequency shift for (a) "Ohmic case" and (b) "constant case."

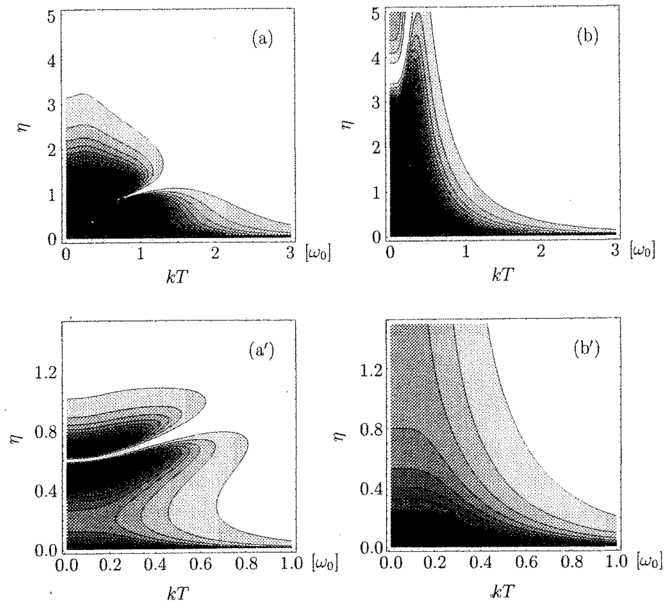

FIG. 5. Temperature- and η-dependence of SNR for (a) "Ohmic case" and (b) "constant case" with $\Omega = 0.10\omega_0$, $\Lambda = 2.0\omega_0$, $\mu = 0.50\omega_0$, and $\Delta/\omega_0 = 0.35$. (a′) and (b′) are enlarged versions of (a) and (b), respectively. Darker grays correspond to larger $R^\beta(\tilde{\Omega})$.

the noise strength η. In Fig. 5, the SNRs $R^\beta(\tilde{\Omega})$ for both cases are shown in the η-T plane. One may realize at first sight that SNR depends deeply on the choice of $J^\beta(\omega)$, namely, on the temperature dependence of the frequency shift σ^β. The temperature dependences of the SNRs are shown in Fig. 6(a) for the "Ohmic case" with $\eta = 0.59$, and in Fig. 6(b) for the "constant case" with $\eta = 3.5$. maximum values are seen at around the temperature $kT \sim 0.3\omega_0$ for both cases, that is, SR occurs. Roughly speaking, these maximum points correspond to the minima of the denominator in the right hand side of Eq. (4.7). One has to notice, however, that this does not occur for all η: for some η it occurs, and for others it does not. And beyond these two possibilities, one can find some strange phenomena. See Fig. 6(a′), where η is chosen as $\eta = 0.65$ for the "ohmic case," and Fig. 6(b′), where $\eta = 4.5$ for the "constant case." There exist temperatures where the system does not respond. We may call this "anti-resonance." It occurs when the frequency shift σ^β coincides with the system frequency $\tilde{\omega}_0$. See the numerator of the amplitude $A^\beta(\tilde{\Omega})$. And see Fig. 6(a″) with $\eta = 0.70$ for the "ohmic case" and Fig. 6(b″) with $\eta = 15$ for the "constant case," where the SNRs have two peaks, i.e., "double resonance." The second maximum comes from the overlapping effect of a negatively decreasing factor ω_R^β beyond its zero point and a positively decreasing Planck distribution. And it does not correspond to a genuine SR. It has no counterpart in classical systems. The behavior of the frequency shift σ^β may

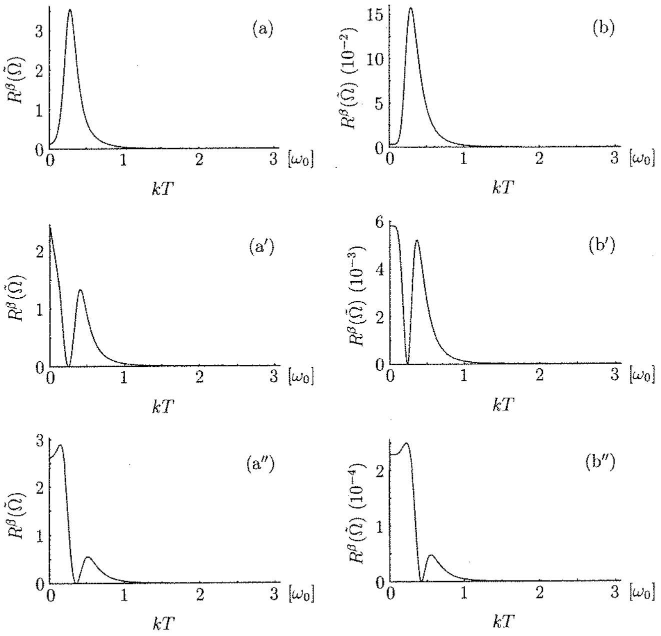

FIG. 6. Temperature-dependence of SNR for "Ohmic case" with (a) $\eta = 0.59$, (a′) $\eta = 0.65$, and (a″) $\eta = 0.70$, and for "constant case" with (b) $\eta = 3.5$, (b′) $\eta = 4.5$, and (b″) $\eta = 15$.

be the key to these phenomena.

V. SUMMARY

QSR in the driven spin-boson system is discussed with quantum white noise introduced through the SLA. The SLA is a framework which can be used to describe the van Hove limit, which ensures the approach of system in a thermal environment to the thermal equilibrium state. SNRs versus noise parameters—temperature T and the noise strength η—are studied with two model spectral functions $J(\omega)$. The occurrence of SR depends on the choice of η. For some η the system does not resonate, for other values it does, and a new phenomenon—anti-resonance and double resonance— is observed. The temperature dependence of the frequency shift of the system σ^β due to the interaction with the environment—Lamb shift—may be the key to these phenomena. In this sense, QSR owes its existence to a quantum effect, which is different from the classical SR, where random force itself is important. To understand this point, this system should be studied in the crossover area between the quantum and classical regimes. This work is now in progress.

It should further be emphasized that the analysis here is from the microscopic view point, not from a semi-phenomenological viewpoint. In the latter there is no criterion which would tell us how to incorporate the damping coefficient γ^β and the frequency shift σ^β into the phenomenological equation properly. Here the damping dynamics is obtained from the fundamental microscopic Hamiltonian underneath the theory.

Finally, let us mention experimental situations for the present analysis.

(1) The physical time is not t but τ. Experimental data should be compared with the theoretical predictions from the analysis in this paper in the macroscopic time τ.

(2) It is very difficult in general to prepare precise quantum mechanical initial conditions experimentally. Fortunately, however, SNR is obtained from the stationary behavior of the system at large times $\tau \gg 1/\gamma^\beta$, and is irrespective of the initial condition.

(3) It is possible to control $J(\omega)$ in the cavity QED experiment. In fact, the life-time of the unstable state of an atom can be successfully controlled by changing the modes of the electromagnetic field, i.e., by changing $J(\omega)$. This means that it may be possible to observe SNRs with different choices of $J(\omega)$ in the cavity QED.

There may be technical difficulties to overcome, but it may be possible to observe experimentally the phenomena predicted here. This would also be an experimental verification of SLA itself.

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APPENDIX

Here we briefly describe SLA from a physical point of view. Introducing a generalized rescaled time as

$$t \mapsto \tau = \lambda^\nu t, \quad \nu > 0, \tag{A1}$$

one has the Tomonaga–Schwinger equation

$$\frac{d}{d\tau} U_I^{(\lambda)}(\tau/\lambda^\nu) = -i\frac{1}{\lambda^{\nu-1}} \sum_\alpha \Big(D_\alpha A_\alpha^\dagger(\tau/\lambda^\nu) + D_\alpha^\dagger A_\alpha(\tau/\lambda^\nu) \Big) U_I^{(\lambda)}(\tau/\lambda^\nu). \tag{A2}$$

We require that the rescaled operators A_α, $A_\alpha^\dagger$ should satisfy the commutation relations with respect to the rescaled time τ. It is easily shown that the possible choices are only of the form $A_\alpha(\tau/\lambda^\nu)/\lambda^{\nu/2}$ and that the non-trivial commutation relation is

$$\left[\frac{1}{\lambda^{\nu/2}}A_{-}(\tau/\lambda^{\nu}), \frac{1}{\lambda^{\nu/2}}A_{-}^{\dagger}(\tau'/\lambda^{\nu})\right]$$
$$= 2\left(\frac{\Delta}{\omega_0}\right)^2\left[J(\omega_0) + i\lambda^{\nu}J'(\omega_0)\frac{\partial}{\partial\tau} + \cdots\right]$$
$$\times \frac{1}{2\pi}\int_{-\omega_0/\lambda^{\nu}}^{\infty} dx\, e^{-ix(\tau-\tau')}, \quad \text{(A3)}$$

while the others vanish. If the first and higher derivatives of the spectral function $J^{(n)}(\omega_0)\,(n = 1, 2, \ldots)$ do not have singularities, one can safely neglect all terms other than $J(\omega_0)$ from the expansion. This corresponds simply to the choice of diagonal singularity, i.e., only the boson mode $\omega_k = \omega_0$ contributes to the damping coefficient in the scaling limit $\lambda \to 0$.

From the above considerations, it is convenient to rewrite Eq. (A2) as

$$\frac{d}{d\tau}U_I^{(\lambda)}(\tau/\lambda^{\nu})$$
$$= -i\lambda^{1-\nu/2}\sum_{\alpha}\left(D_{\alpha}\frac{1}{\lambda^{\nu/2}}A_{\alpha}^{\dagger}(\tau/\lambda^{\nu})\right.$$
$$\left. + D_{\alpha}^{\dagger}\frac{1}{\lambda^{\nu/2}}A_{\alpha}(\tau/\lambda^{\nu})\right)U_I^{(\lambda)}(\tau/\lambda^{\nu}). \quad \text{(A4)}$$

Thus, one can see that, in the limit $\lambda \to 0$, (1) the right hand side of Eq. (A4) vanishes in the case of "under" SLA ($\nu < 2$), while (2) it diverges in the case of "over" SLA ($\nu > 2$), and (3) it has a formal limit in the case of "critical" SLA ($\nu = 2$). Therefore the only meaningful result occurs in the $\nu = 2$ case.

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